Overview of FFAG

FFAG School
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KURNS

FFAG (Fixed Field Alternating Gradient)

固定磁場 強収束 加速器

Cyclotron　　　　Synchrotron

FFAG

unless otherwise specified

FFAG synchrotron

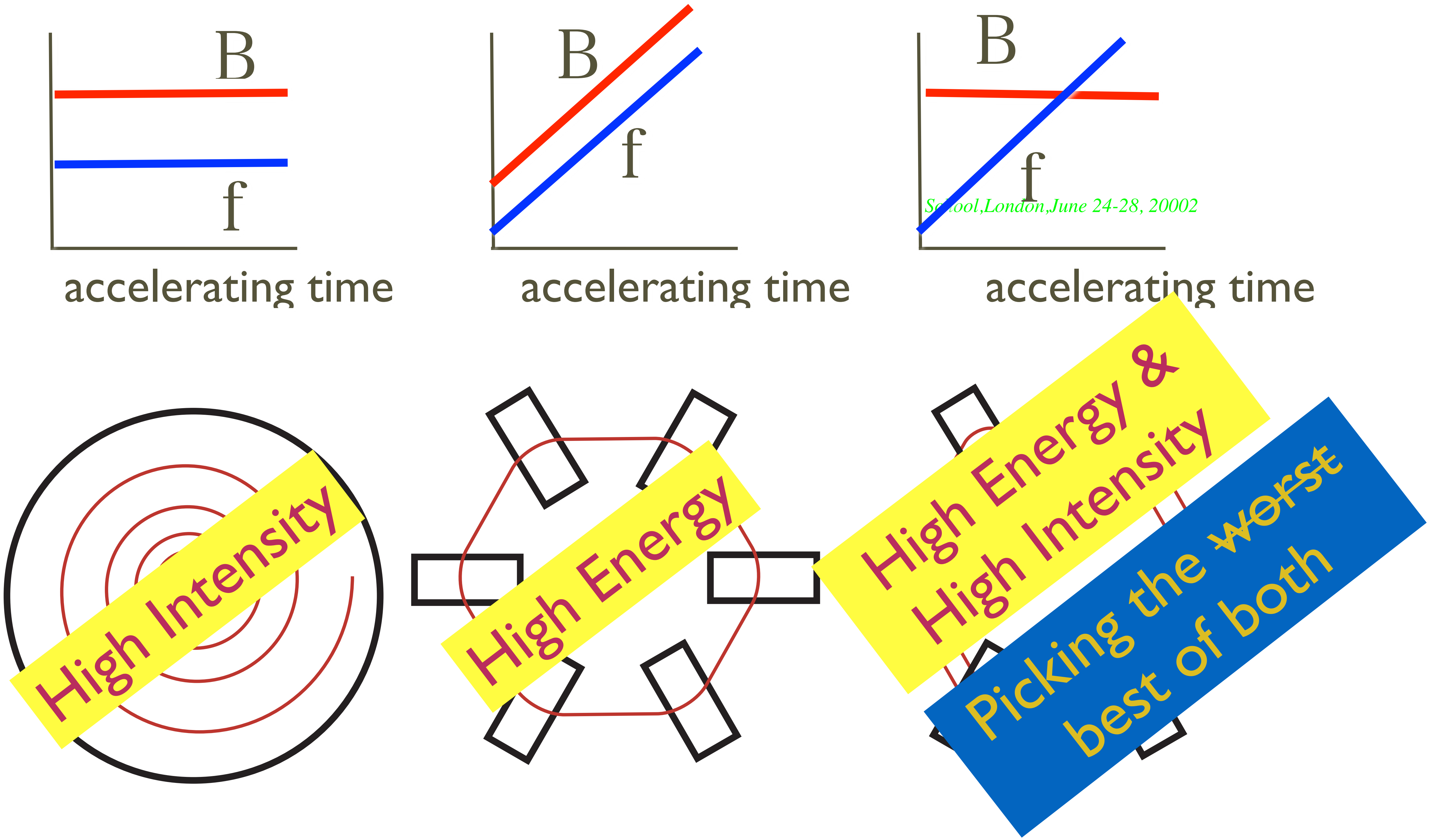
FFAG cyclotron

FFAG betatron

Characteristics of Circular Accelerators

	Cyclotron	Synchrotron	scaling FFAG (synchrotron)
Revolution time	Constant	Variable (except ultra relativistic)	Variable (expect special condition)
Orbit radius	Variable	Constant	Variable
Transverse focusing (betatron tune)	Variable	Constant	Constant
B field during acceleration	static	dynamic	static

cf *FFAG Optics and Dynamics .FFAG School, Fukuoka, Japan September 2015, Dr. Suzie Sheehy, University of Oxford and ASTeC/STFC/RAL*



Cyclotron
*isochronous

Synchrotron
*const. closed orbit
(varying mag. field)

FFAG
*varying closed orbit
(const. mag. field)

Suitable for mass transport,
but not for high floor
Cyclotron

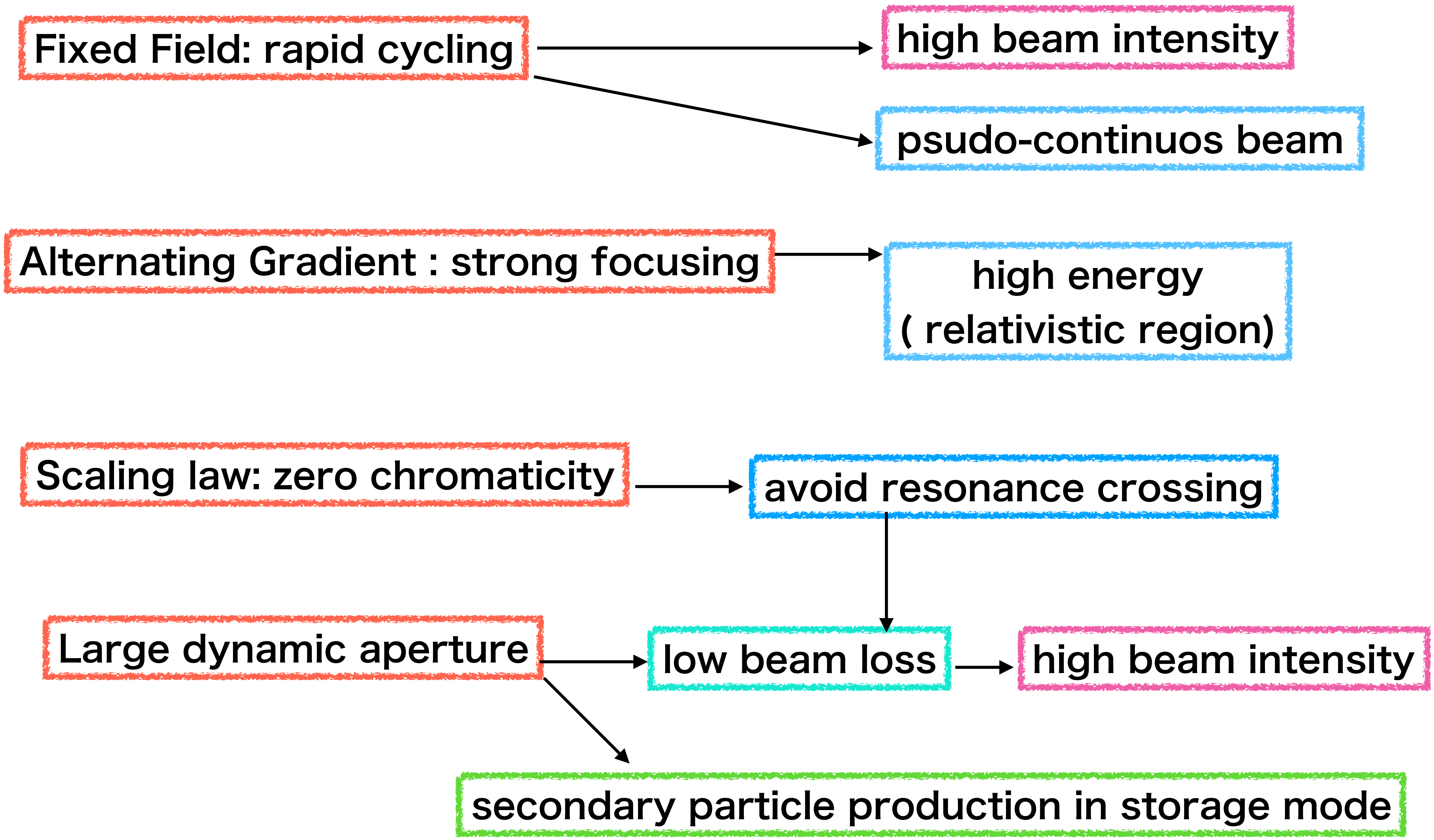
Rapid operated elevator
FFAG

Suitable for high floor,
but not mass transport
Synchrotron

What’s FFAG?

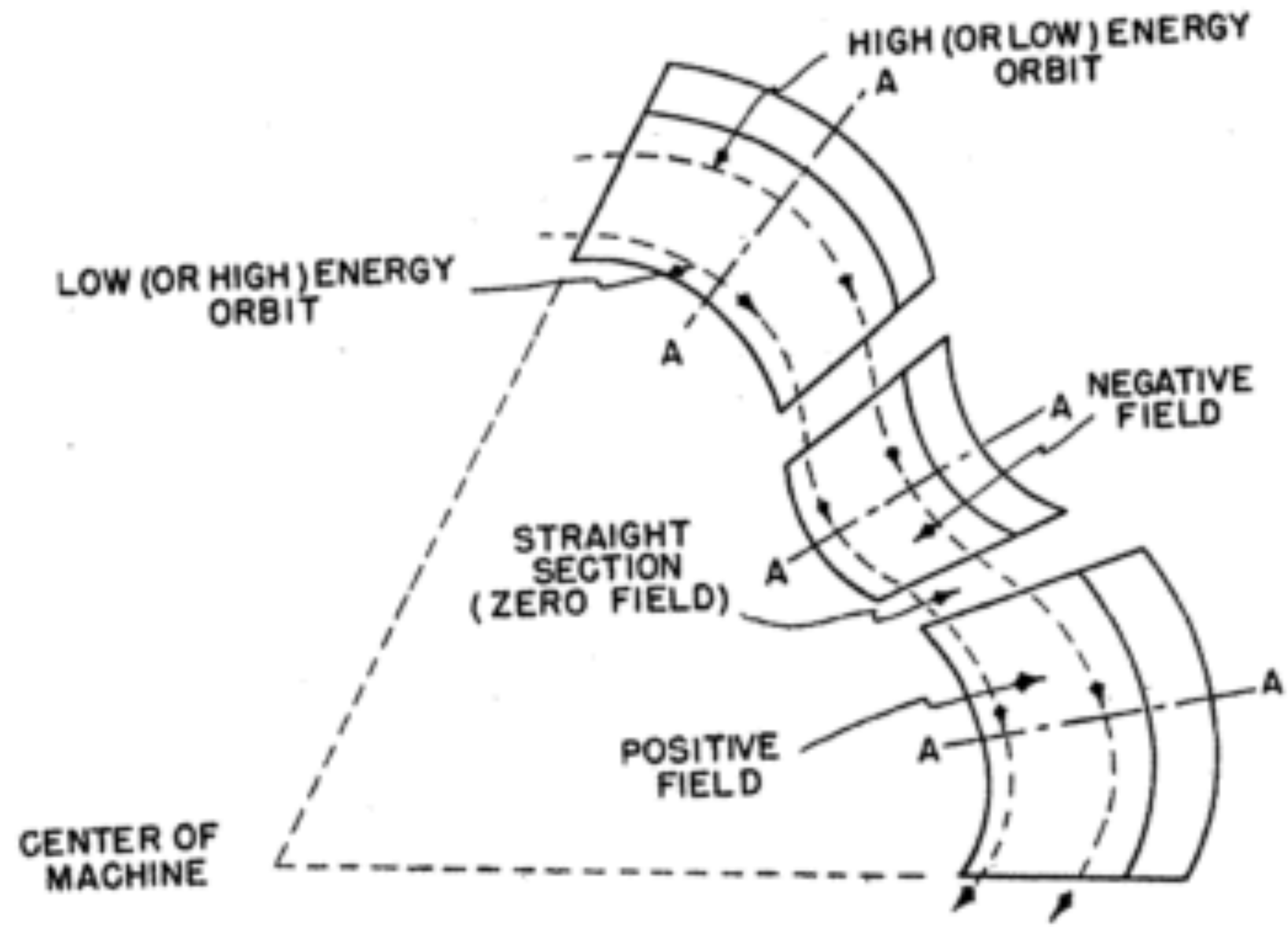
1. FFAG (Fixed Field Alternating-Gradient) :scheme of transverse beam focusing.
2. B field is constant in time.
3. B field provides focusing force in both horizontal and vertical directions.
4. B field configuration is arranged so that the beam experiences alternating gradient.
5. Consequently, the beam gets strong focusing force in both horizontal and vertical.
6. The chromaticity is “zero”.
7. High energy and high intensity can be realized at the same time.
8. The concept was proposed by Tihiro Ohkawa (Japan), Andrei Kolomenskij (CCCP) and Keith Simon (USA) independently in mid 1950s.
9. Proof of principle machines were built at MURA (Midwestern University Research Association), which is predecessor of Fermi Lab.
10. It was not practically used for a long time because of hardware difficulties.
11. Yoshiharu Mori and his group has overcome them. The path of practical use has been opened.

keywords



Fixed Field: rapid cycling
high beam intensity
psudo-continuos beam
Alternating Gradient : strong focusing
high energy
(relativistic region)
Scaling law: zero chromaticity
avoid resonance crossing
Large dynamic aperture
low beam loss
high beam intensity
secondary particle production in storage mode

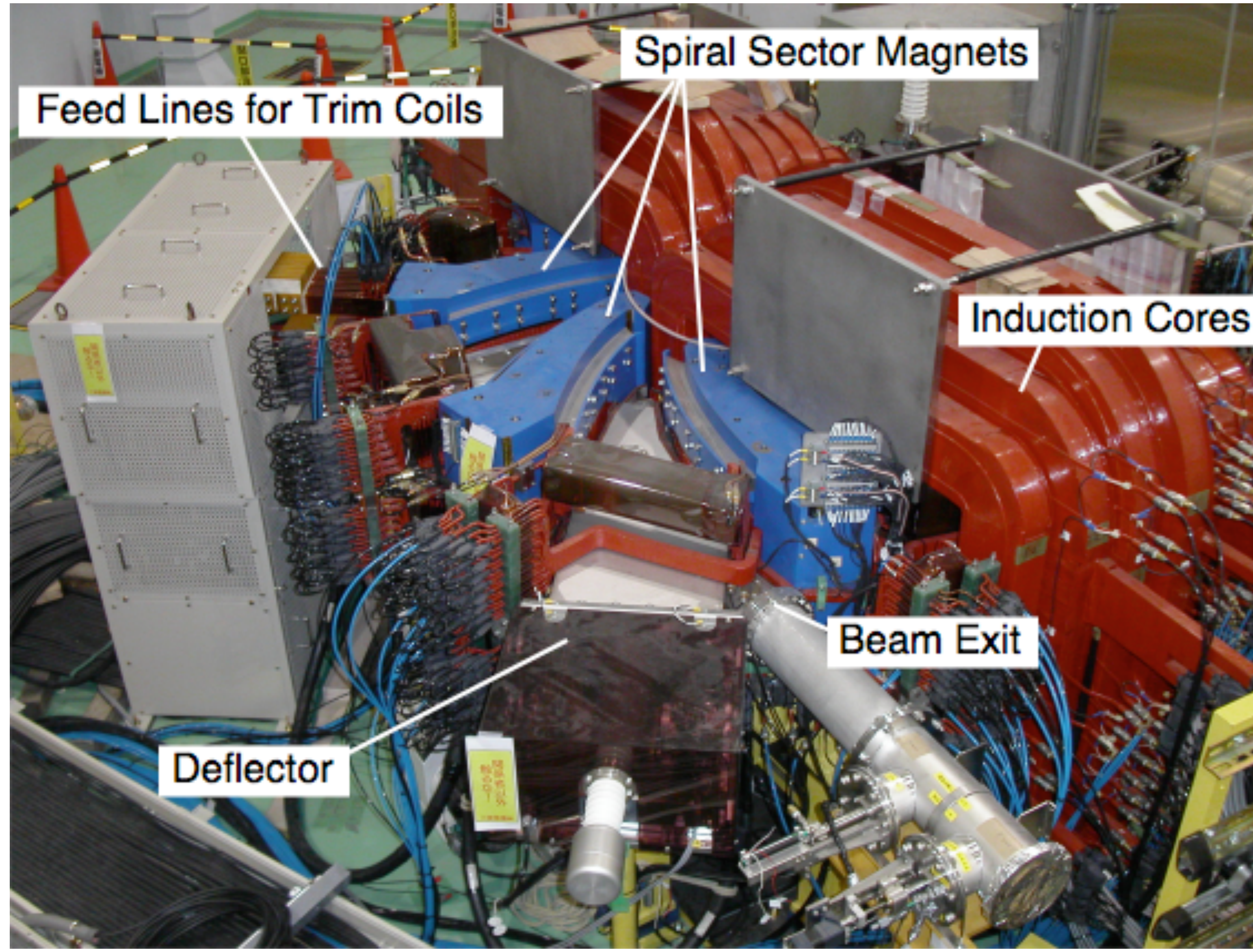
radial sector



negative k: the orbit go inner during acceleration

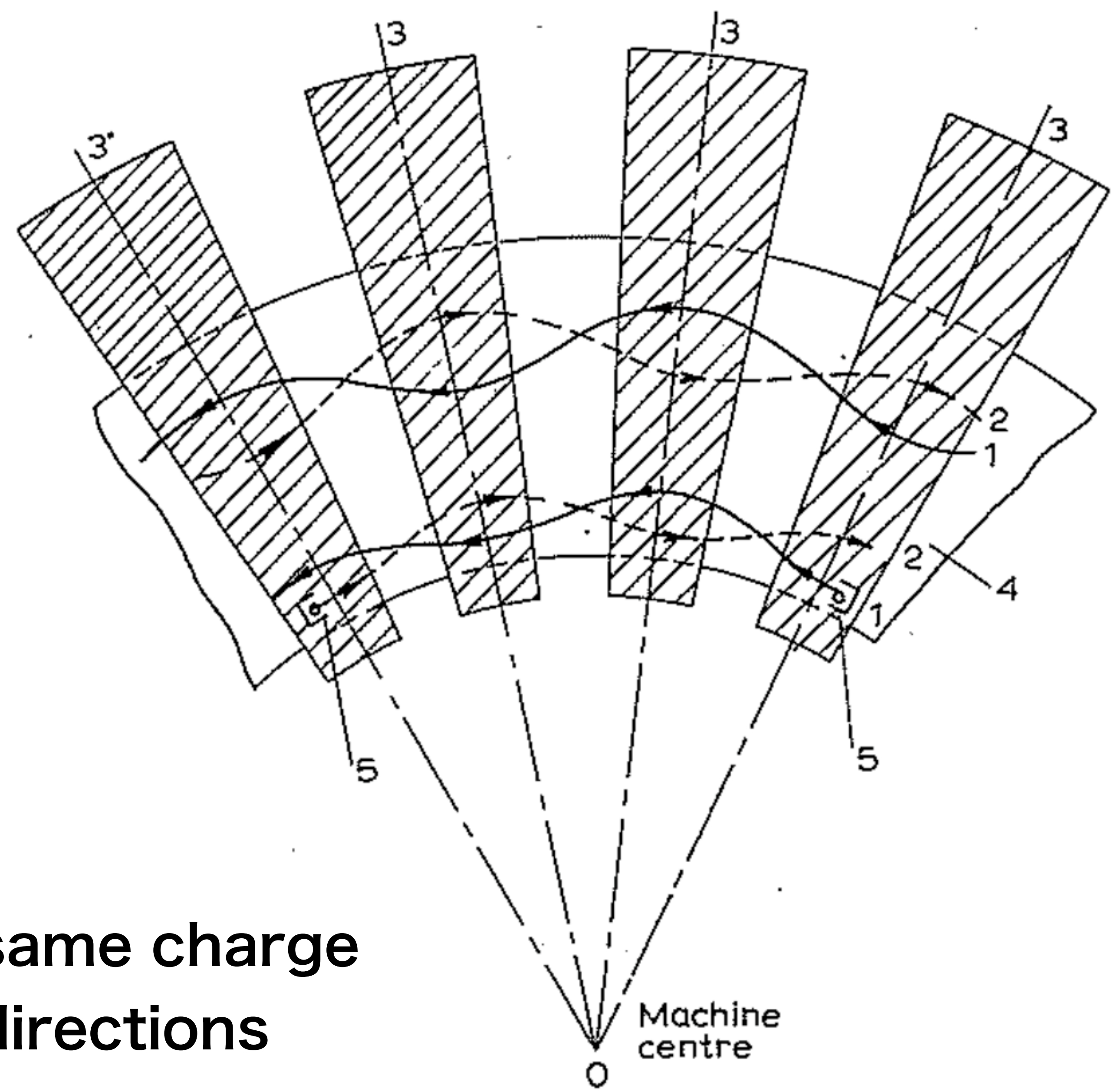
Spiral sector

KURNS Ion-beta



edge focusing: $\Delta p_y = -eB_y \tan\zeta\, y$

Kolomenskij two beam FFAG (Synchro crash)



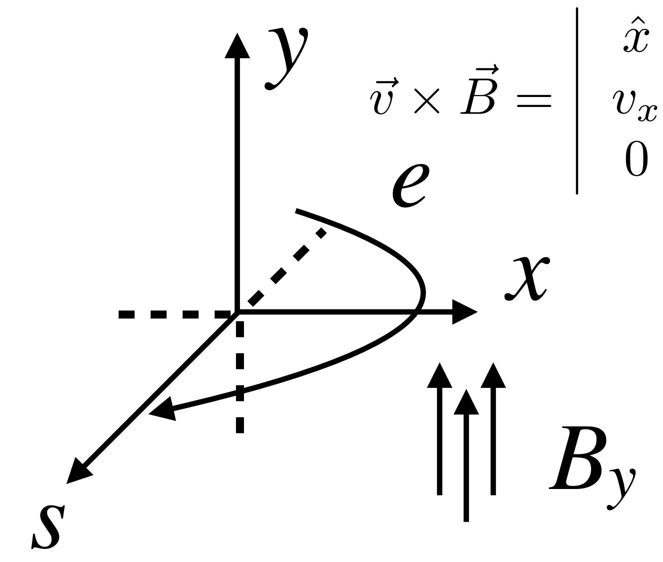
Beams : the same charge
opposite directions

OUTLINE

- Motion of changed particles in uniform magnetic filed.
- What is the betatron tune.
- Tranfere matrix.
- Stability criteria.
- Strong focusing and FFAG

Motion of changed particles in uniform magnetic filed.

$$\dot{\vec{p}} = \vec{F} \qquad \vec{p} = m\vec{v}\gamma$$



$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{s} \\ v_x & v_y & v_s \\ 0 & B_y & 0 \end{vmatrix}$$

$$\dot{\vec{p}} = m\dot{\vec{v}}\gamma + m\vec{v}\dot{\gamma} \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\vec{v} \cdot \vec{F} = \vec{v} \cdot (e\vec{v} \times \vec{B}) = 0 \Rightarrow \dot{\gamma} = 0$$

$$m\dot{\vec{v}}\gamma = e\vec{v} \times \vec{B}$$

$$\begin{pmatrix} \dot{v_x} \\ \dot{v_y} \\ \dot{v_s} \end{pmatrix} = \frac{eB_y}{m\gamma} \begin{pmatrix} -v_s \\ 0 \\ v_x \end{pmatrix}$$

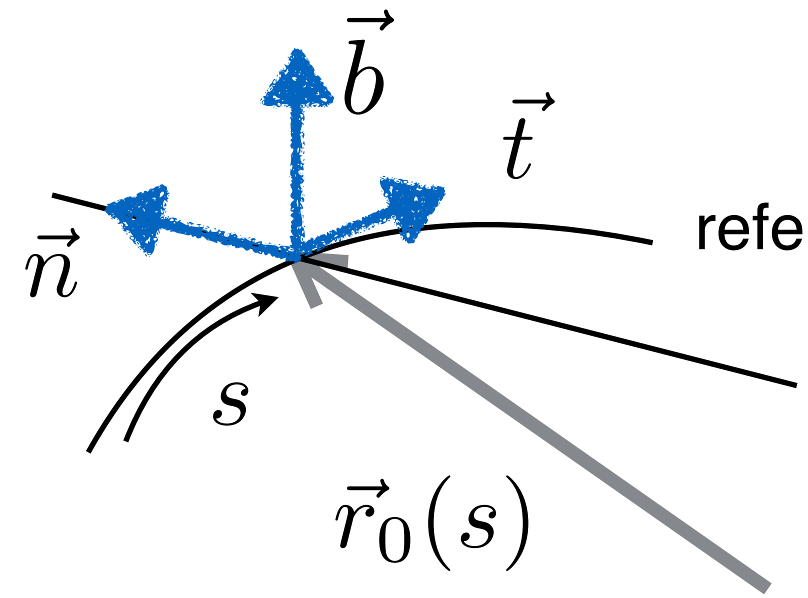
$\frac{eB_y}{m\gamma} \rightarrow \omega$

$$\begin{aligned} v_x &= v\cos(\omega t + \phi) \\ v_s &= v\sin(\omega t + \phi) \\ v &= \rho\omega \\ &= \frac{eB\rho}{\gamma m} \\ p &= eB\rho \end{aligned}$$

Frenet - Serret Coordinate System

unit vectors

$$\vec{t}(s) = \text{tangential}$$
$$\vec{n}(s) = \text{normal}$$
$$\vec{b}(s) = \text{binormal}$$



$$\vec{r} = \vec{r}_0(s) + x\vec{n}(s) + y\vec{b}(s)$$

Position of a particle can be determined by using the coordinate system on a curve so-called reference orbit in the accelerator.

$$\frac{d\vec{r}_0}{ds} = \vec{t}$$

$$\frac{d\vec{t}}{ds} = -\frac{\vec{n}}{\rho} \qquad \frac{d\vec{n}}{ds} = \tau\vec{b} + \frac{\vec{t}}{\rho} \qquad \frac{d\vec{b}}{ds} = -\tau\vec{n}$$

$$(\vec{n}, \vec{b}, \vec{t}) = (\hat{x}, \hat{y}, \hat{s})$$

Equation of motion

$$x'' + \left[\frac{1}{\rho^2} + \frac{1}{B\rho}\frac{\partial B_y}{\partial x}\right] x = 0$$

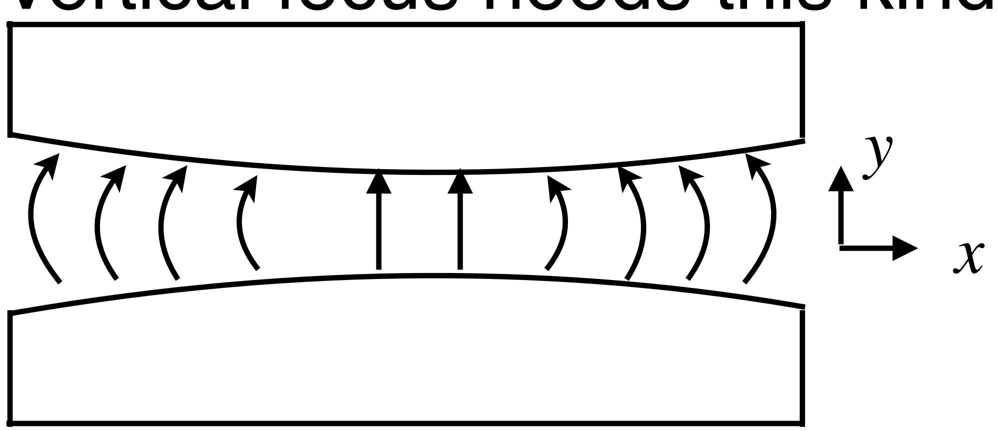
will be derived in "appendix"

$$y'' - \frac{1}{B\rho}\frac{\partial B_y}{\partial x} y = 0$$

In order to solve these equations, we want to know function B.

In this case, we only need to know the derivative of By w.r.t. radial direction in the mid-plane ($z = 0$) and on the reference orbit ($r = \rho$).

Vertical focus needs this kind of shape



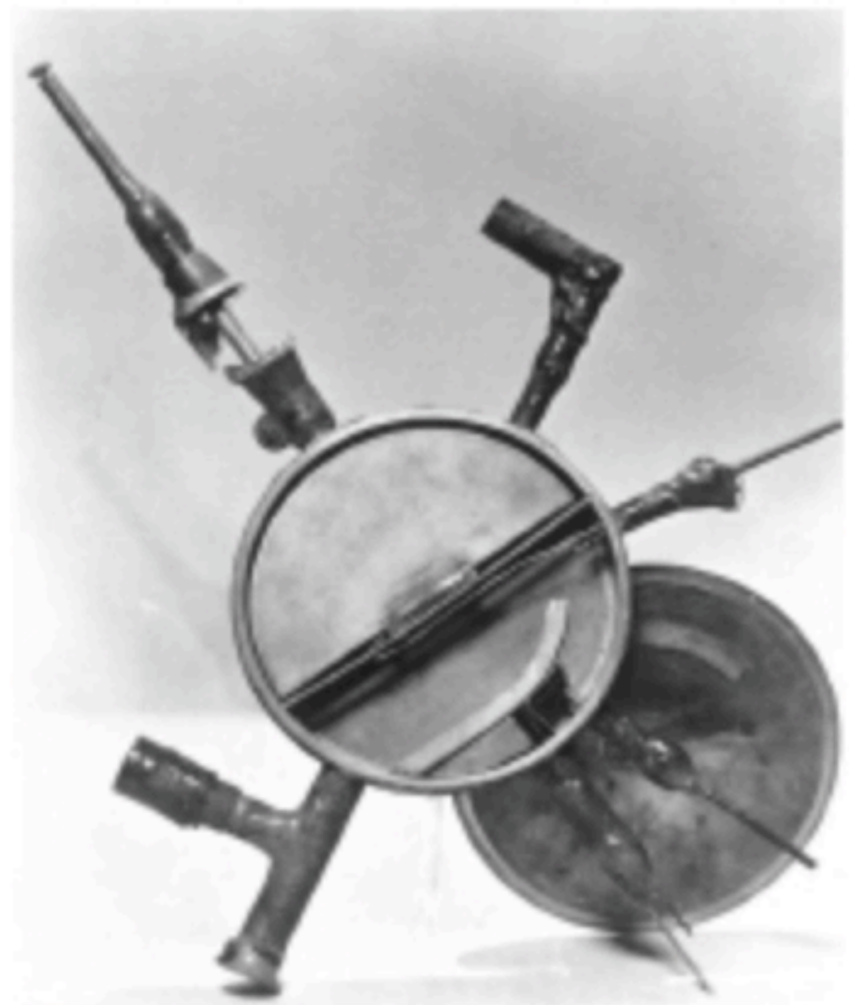
$$\vec{B} = \vec{B}(x, y)$$

$$\frac{\partial B_y}{\partial x}\frac{\rho}{B_0} = -n$$

$$(r = \rho + x)$$

Cyclotron

E. Lawrence



4” proof of principle machine accelerated hydrogen molecule up to 80 keV

$$x'' + \left[\frac{1}{\rho^2} + \frac{1}{B\rho}\frac{\partial B_y}{\partial x}\right]x = 0 \qquad y'' - \frac{1}{B\rho}\frac{\partial B_y}{\partial x}y = 0 \qquad \frac{\partial B_y}{\partial x}\frac{\rho}{B_0} = -n$$

$$x'' + \frac{1}{\rho^2}(1-n)x = 0$$

$$y'' + \frac{n}{\rho^2}y = 0$$

These are well known equations of the harmonic oscillator。

Condition for stable solutions in both x and y.

$$0 < n < 1$$

Betatron tune

Solution for the vertical direction.

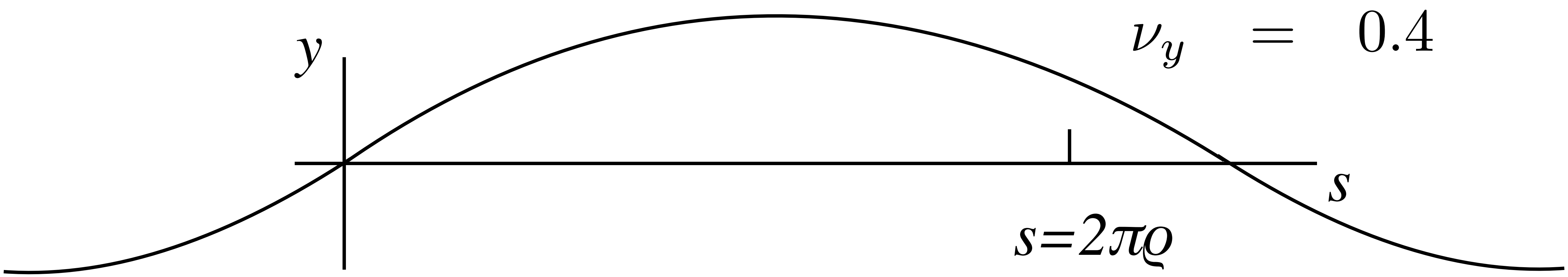
$$y(s) = C\cos\left(\frac{\sqrt{n}}{\rho}s + \phi_0\right)$$

This oscillation is called "betatron oscillation".
The phase advance during one turn divided by 2π is called ν_y "betatron tune", or simply "tune".

$$y(s_0 + 2\pi\rho) = C\cos\left(\frac{\sqrt{n}}{\rho}2\pi\rho + \phi_0\right) = C\cos\left(2\pi\nu_y + \phi_0\right)$$

$$\nu_y = \sqrt{n}$$

$$n = 0.16$$
$$\nu_y = 0.4$$



Tune is one of the most important parameter in the accelerator.

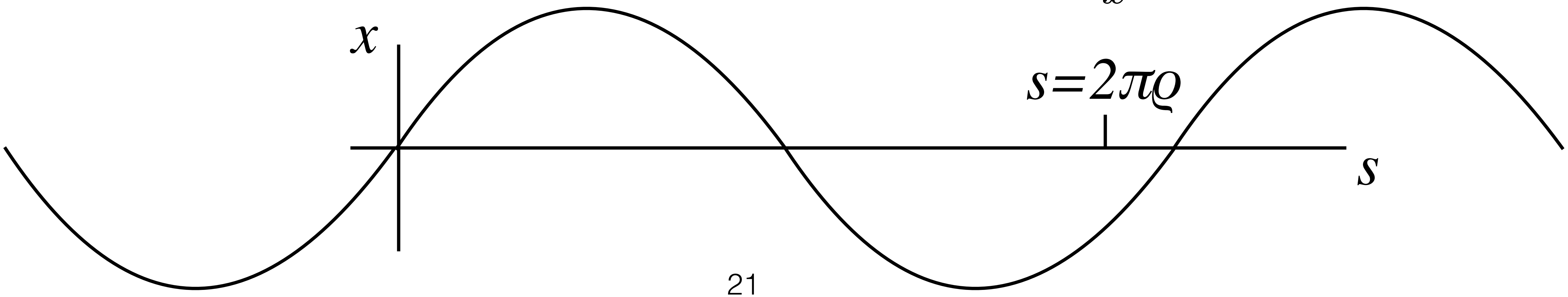
For horizontal similarly

$$x(s) = C\cos\left(\frac{\sqrt{1-n}}{\rho}s + \phi_0\right)$$

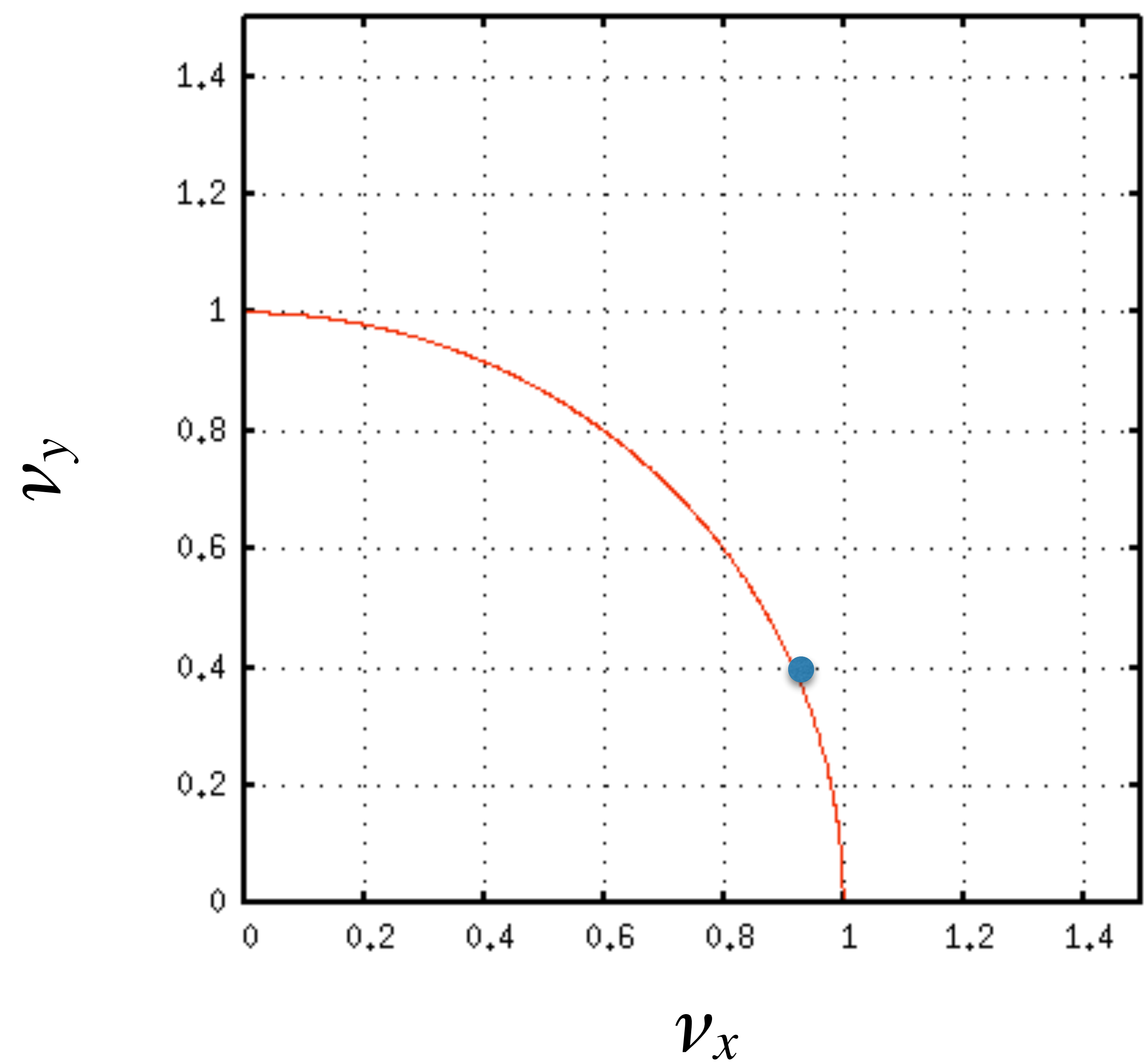
$$\begin{aligned} x(s_0 + 2\pi\rho) &= C\cos\left(\frac{\sqrt{1-n}}{\rho}2\pi\rho + \phi_0\right) \\ &= C\cos\left(2\pi\nu_x + \phi_0\right) \end{aligned}$$

$$\nu_x = \sqrt{1-n}$$

$$\begin{aligned} n &= 0.16 \\ \nu_x &= 0.917 \end{aligned}$$



TUNE DIAGRAM



$$\nu_x = \sqrt{1-n}$$

$$\nu_y = \sqrt{n}$$

$$0 < n < 1$$

Appendix

Relativistic Lagrangian

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}}$$

with EM filed

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\phi + e\vec{A}\cdot\vec{v}$$

Hamiltonian

$$H = c\sqrt{(\vec{p}-e\vec{A})^2 + m^2c^2} + e\phi$$

Hamiltonian in the accelerator coordinate system

$$\tilde{H}(x, y, s, p_x, p_y, -H, s)$$
$$= -(1+hx)\left[\frac{1}{c^2}(H-e\phi)^2 - m^2c^2 - (p_x - eA_x)^2 - (p_y - eA_y)^2\right]^{\frac{1}{2}} - eA$$

Linear motion in the ring

Hamiltonian

$$\tilde{H} = -(1+hx)\left[\frac{(H-e\phi)^2}{c^2} - m^2c^2 - (p_x - eA_x)^2 - (p_y - eA_y)^2\right]^{\frac{1}{2}} - eA_s$$

$$h = \frac{1}{\rho}$$

Consider only magnetic field with no time dependence, no electric field.

$$\tilde{H} = -(1+hx)\left[\frac{H^2 - m^2c^4}{c^2} - (p_x - eA_x)^2 - (p_y - eA_y)^2\right]^{\frac{1}{2}} - eA_s$$

$$\tilde{H} = -(1+hx)\left[p^2 - (p_x - eA_x)^2 - (p_y - eA_y)^2\right]^{\frac{1}{2}} - eA_s$$

$$\tilde{H} = -(1+hx)\left[p_0^2(1+\delta) - (p_x - eA_x)^2 - (p_y - eA_y)^2\right]^{\frac{1}{2}} - eA_s$$

Only for on-momentum particle. $\delta = 0$

$$\tilde{H} = -(1+hx)\left[p_0^2 - (p_x - eA_x)^2 - (p_y - eA_y)^2\right]^{\frac{1}{2}} - eA_s$$

Field component in the beam axis is 0 i.e. $Bs=0$. Then one can set as $Ax=0, Ay=0$

$$\tilde{H} = -(1+hx)\left[p_0^2 - p_x^2 - p_y^2\right]^{\frac{1}{2}} - eA_s$$

$$\frac{\tilde{H}}{p_0} \to H \qquad \frac{p_x}{p_0} \to p_x \qquad \frac{p_y}{p_0} \to p_y \qquad \frac{eA_s}{p_0} \to a_s$$

$$H = -(1+hx)\sqrt{1 - p_x^2 - p_y^2} - a_s$$

Applying paraxial approximation, and neglecting higher order terms from $p_x hx, p_y hx$

$$H \sim \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 - (1+hx) - a_s$$

$$\begin{aligned} a_s &= (1+hx)\boldsymbol{a}\cdot\hat{s} \\ &= (1+hx)a_3 \end{aligned}$$

Vector potential A_3 can be expanded as

$$A_3 \quad = \quad -B_0\left(x - \frac{hx^2}{2(1+hx)}\right) - B_1\left(\frac{1}{2}(x^2 - y^2) + \cdots\right) + \cdots$$

where B_0 and B_1 are multipole components on the mid-plane.

$$B_y(x,0,s) = B_0 + B_1\frac{x}{1!} + B_2\frac{x^2}{2!} + \cdots$$

$$B_x(x,0,s) = 0, \qquad\qquad B_s(x,0,s) = 0$$

cf C.J. Gardner, Particle Accelerators, 1991, Vol. 35 pp. 215 - 226

Hamiltonian becomes

$$H \sim \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 - (1+hx)(1+a_3)$$

Then we use $\frac{B_0}{B\rho} = h$ and pick terms of x and y up to 2nd order.

$$H_{\rm lin} = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}h^2x^2 + \frac{B_1}{2B\rho}(x^2 - y^2)$$

Equation of motion

$$\begin{aligned} x' &= \frac{\partial H_{\rm lin}}{\partial p_x} \\ &= p_x \\ p_x' &= -\frac{\partial H_{\rm lin}}{\partial x} \\ &= -h^2 x - \frac{B_1}{B\rho}x \end{aligned}$$

$$\begin{aligned} y' &= \frac{\partial H_{\rm lin}}{\partial p_y} \\ &= p_y \\ p_y' &= -\frac{\partial H_{\rm lin}}{\partial y} \\ &= \frac{B_1}{B\rho}y \end{aligned}$$

summary of appendix

Hamiltonian in an accelerator

$$\tilde{H} = -(1+hx)\left[\frac{(H-e\phi)^2}{c^2} - m^2c^2 - (p_x - eA_x)^2 - (p_y - eA_y)^2\right]^{\frac{1}{2}} - eA_s$$

linear approximation

$$H_{\rm lin} = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}h^2x^2 + \frac{B_1}{2B\rho}(x^2 - y^2)$$

canonical equation

Hill's equation

$$\begin{aligned} x'' + \left(h^2 + \frac{B_1}{B\rho}\right)x &= 0 \\ x'' + K_x(s)x &= 0 \end{aligned} \qquad \begin{aligned} y'' - \frac{B_1}{B\rho}y &= 0 \\ y'' + K_y(s)y &= 0 \end{aligned}$$

Transfer matrix

So far, we assume the magnetic field does not have azimuthal dependence like a cyclotron. However, rings like a synchrotron have straight sections. For these rings, simple stability criteria $0< n < 1$ does not work.

Now we introduce a transfer matrix. First we treat a vertical case.

$$y(s) \quad = \quad C\cos\left(\frac{\sqrt{n}}{\rho}s + \phi_0\right)$$

$$\Downarrow$$

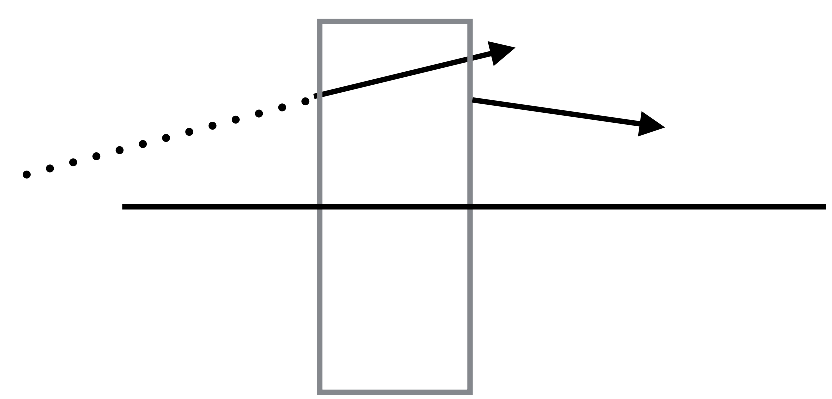
$$y(s) = C_1 \cos\frac{\sqrt{n}}{\rho}s + C_2 \sin\frac{\sqrt{n}}{\rho}s$$

$$y(s) = C_1 \cos \frac{\sqrt{n}}{\rho} s + C_2 \sin \frac{\sqrt{n}}{\rho} s$$

$$y_0 \equiv y(0) = C_1$$

$$y'(s) = -y_0 \frac{\sqrt{n}}{\rho} \sin \frac{\sqrt{n}}{\rho} s + C_2 \frac{\sqrt{n}}{\rho} \cos \frac{\sqrt{n}}{\rho} s$$

$$\begin{aligned} y'_0 &\equiv y'(0) = C_2 \frac{\sqrt{n}}{\rho} \\ C_2 &= \frac{\rho}{\sqrt{n}} y'_0 \end{aligned}$$

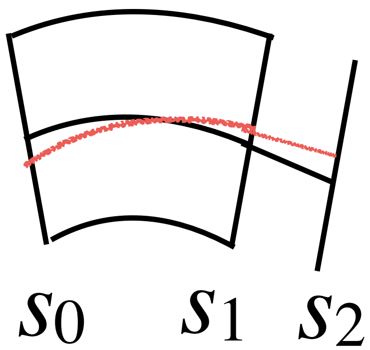


in a matrix form

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} \cos \frac{\sqrt{n}}{\rho} s & \frac{\rho}{\sqrt{n}} \sin \frac{\sqrt{n}}{\rho} s \\ -\frac{\sqrt{n}}{\rho} \sin \frac{\sqrt{n}}{\rho} s & \cos \frac{\sqrt{n}}{\rho} s \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$$

Similarly, in horizontal

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos\frac{\sqrt{1-n}}{\rho}s & \frac{\rho}{\sqrt{1-n}}\sin\frac{\sqrt{1-n}}{\rho}s \\ -\frac{\sqrt{1-n}}{\rho}\sin\frac{\sqrt{1-n}}{\rho}s & \cos\frac{\sqrt{1-n}}{\rho}s \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



unit cell

$$\begin{aligned} \vec{x_1} &= M_x(s_1|s_0)\vec{x_0} \\ \vec{x_2} &= M_x(s_2|s_1)\vec{x_1} \\ \vec{x_2} &= M_x(s_2|s_1)M_x(s_1|s_0)\vec{x_0} \\ \vec{x_2} &= M_x(s_2|s_0)\vec{x_0} \end{aligned}$$

$$\begin{aligned} \vec{y_1} &= M_y(s_1|s_0)\vec{y_0} \\ \vec{y_2} &= M_y(s_2|s_1)\vec{y_1} \\ \vec{y_2} &= M_y(s_2|s_1)M_y(s_1|s_0)\vec{y_0} \\ \vec{y_2} &= M_y(s_2|s_0)\vec{y_0} \end{aligned}$$

for straight section

$$\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \rightarrow \begin{pmatrix} x_0 + x'_0\Delta s \\ x'_0 \end{pmatrix} = \begin{pmatrix} 1 & \Delta s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Stability criteria

For example, suppose the ring has 12-fold symmetry and acceleration needs 1E6 turns. One can confirm if this ring is stable multiplying the transfer matrix for one period 12E6 times.

$$\vec{x}_f = M_x^k \vec{x}_i \qquad k = 12\text{E}6$$

But it’s tiresome. (Actually, easy for computers)

Stability criteria

$$|\,\mathrm{Tr}\, M\,| \le 2$$

Let eigen value of M λ,

$$\left| \begin{array}{cc} m_{11}-\lambda & m_{12} \\ m_{21} & m_{22}-\lambda \end{array} \right| = 0$$

$$\lambda^2 - (m_{11}+m_{22})\lambda + (m_{11}m_{22} - m_{12}m_{21}) = 0$$

Let eigen vectors $\begin{pmatrix} \xi \\ \xi' \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix}$

From the e.o.m.

$$\frac{d^2\xi}{ds^2} + \frac{n}{\rho^2}\xi = 0 \quad \xrightarrow{\times\eta} \quad \eta\frac{d^2\xi}{ds^2} + \frac{n}{\rho^2}\xi\eta = 0$$

$$\frac{d^2\eta}{ds^2} + \frac{n}{\rho^2}\eta = 0 \quad \xrightarrow{\times\xi} \quad -)\ \xi\frac{d^2\eta}{ds^2} + \frac{n}{\rho^2}\xi\eta = 0$$

$$\eta\frac{d^2\xi}{ds^2} - \xi\frac{d^2\eta}{ds^2} = 0$$

$$\frac{d}{ds}\left(\eta\frac{d\xi}{ds} - \xi\frac{d\eta}{ds}\right) = 0$$

constant

$$\eta_1\xi_1' - \xi_1\eta_1' = \eta_0\xi_0' - \xi_0\eta_0'$$

$$\eta_1 \xi_1' - \xi_1 \eta_1' = \eta_0 \xi_0' - \xi_0 \eta_0'$$

$$(m_{11}\eta_0 + m_{12}\eta_0')(m_{21}\xi_0 + m_{22}\xi_0') - (m_{11}\xi_0 + m_{12}\xi_0')(m_{21}\eta_0 + m_{22}\eta_0') = \eta_0 \xi_0' - \xi_0 \eta_0'$$

$$\begin{aligned}
m_{11}\eta_0 m_{21}\xi_0 + m_{11}\eta_0 m_{22}\xi_0' + m_{12}\eta_0' m_{21}\xi_0 + m_{12}\eta_0' m_{22}\xi_0' \\
- m_{11}\xi_0 m_{21}\eta_0 - m_{11}\xi_0 m_{22}\eta_0' - m_{12}\xi_0' m_{21}\eta_0 - m_{12}\xi_0' m_{22}\eta_0' \\
= m_{11}m_{22}(\eta_0\xi_0' - \xi_0\eta_0') - m_{12}m_{21}(\eta_0\xi_0' - \xi_0\eta_0') \\
= (m_{11}m_{22} - m_{12}m_{21})(\eta_0\xi_0' - \xi_0\eta_0') \\
= \eta_0\xi_0' - \xi_0\eta_0'
\end{aligned}$$

$$m_{11}m_{22} - m_{12}m_{21} = 1$$

$$\lambda^2 - (m_{11} + m_{22})\lambda + (m_{11}m_{22} - m_{12}m_{21}) = 0$$

$$\lambda^2 - (m_{11} + m_{22})\lambda + 1 = 0$$

$$\lambda^2 - (m_{11} + m_{22})\lambda + 1 = 0$$

$$\begin{aligned}\lambda_a &= \frac{1}{2}(m_{11} + m_{22}) + \frac{1}{2}\sqrt{(m_{11} + m_{22})^2 - 4} \\ \lambda_b &= \frac{1}{2}(m_{11} + m_{22}) - \frac{1}{2}\sqrt{(m_{11} + m_{22})^2 - 4}\end{aligned}$$

$$\lambda_a \lambda_b = 1 \qquad \lambda_a + \lambda_b = m_{11} + m_{22}$$

$$\begin{aligned}\lambda_a &= e^{w} \\ \lambda_b &= e^{-w}\end{aligned}$$

Arbitrary initial vector $\vec{y}_0$ can be expressed using the linear combination of eigenvectors $\vec{\xi}_0, \vec{\eta}_0$

$$\vec{y}_0 = A\vec{\xi}_0 + B\vec{\eta}_0$$

After passing thru this cell N times,

$$\vec{y} = M^N \vec{y}_0 = Ae^{Nw}\vec{\xi}_0 + Be^{-Nw}\vec{\eta}_0$$

if w is real number, e^{Nw} or e^{-Nw} becomes infinity when N→∞

Therefore w should be pure imaginary,

$$w = i\mu$$

$$\begin{aligned} \lambda_a &= e^{i\mu} \\ \lambda_b &= e^{-i\mu} \\ \lambda_a + \lambda_b &= e^{i\mu} + e^{-i\mu} \\ &= 2\cos\mu \end{aligned} \qquad \lambda_a + \lambda_b = m_{11} + m_{22}$$

$$|\mathrm{Tr}\ M| \le 2$$

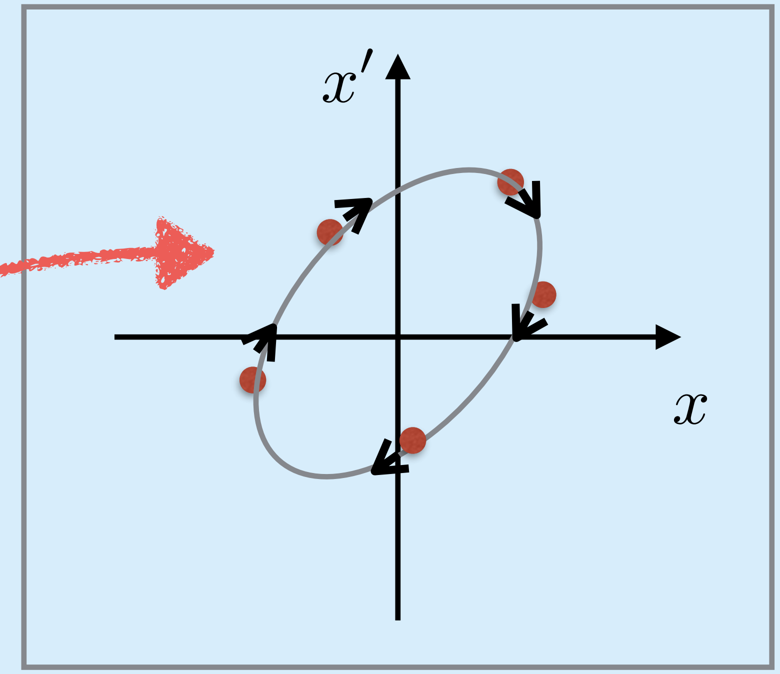
Summary, so far

Equations of motion

$$x'' + \left[\frac{1}{\rho^2} + \frac{1}{B\rho}\frac{\partial B_y}{\partial x}\right] x = 0$$

$$y'' - \frac{1}{B\rho}\frac{\partial B_y}{\partial x} y = 0$$

transfer points in the phase space



phase space

transfer matrix

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos\frac{\sqrt{1-n}}{\rho}s & \frac{\rho}{\sqrt{1-n}}\sin\frac{\sqrt{1-n}}{\rho}s \\ -\frac{\sqrt{1-n}}{\rho}\sin\frac{\sqrt{1-n}}{\rho}s & \cos\frac{\sqrt{1-n}}{\rho}s \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

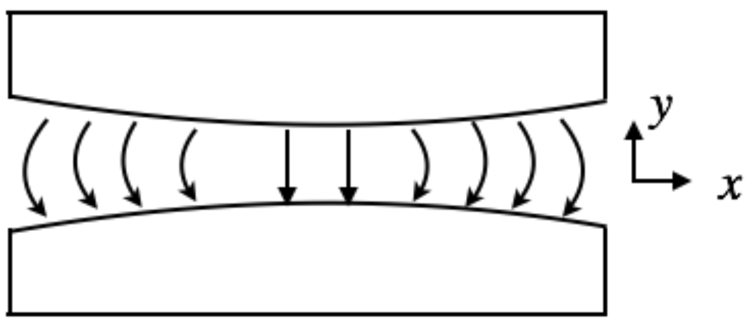
$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} \cos\frac{\sqrt{n}}{\rho}s & \frac{\rho}{\sqrt{n}}\sin\frac{\sqrt{n}}{\rho}s \\ -\frac{\sqrt{n}}{\rho}\sin\frac{\sqrt{n}}{\rho}s & \cos\frac{\sqrt{n}}{\rho}s \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$$

stability criteria (ignoring resonances and higher order effects)

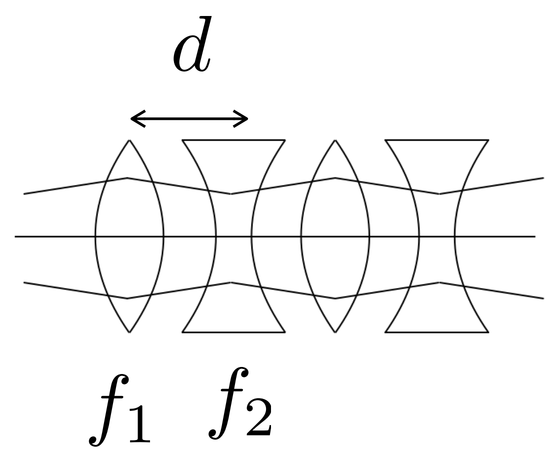
$$|\mathrm{Tr}\ M| \leq 2$$

Strong (AG) focusing and Courant Snyder variables

Discovery of the Principle of Strong Focusing



$$\vec{B} = \vec{B}(x, y)$$



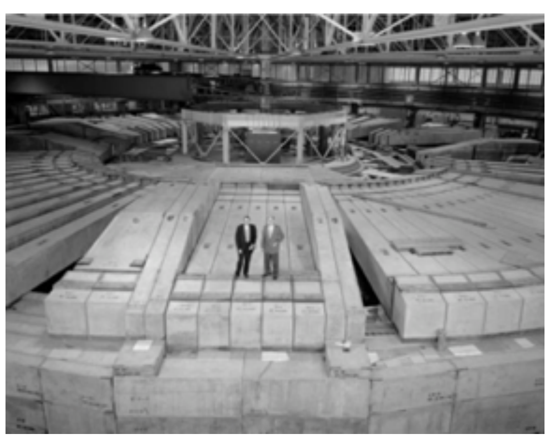
From an analogy of ordinary optics,

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

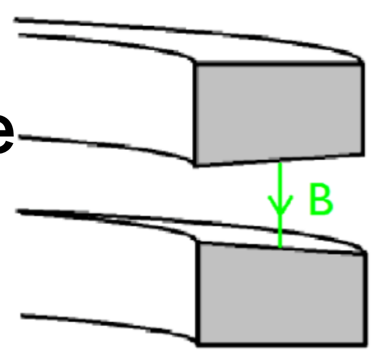
$$f_2 = -f_1$$

$$F = \frac{f^2}{d}$$

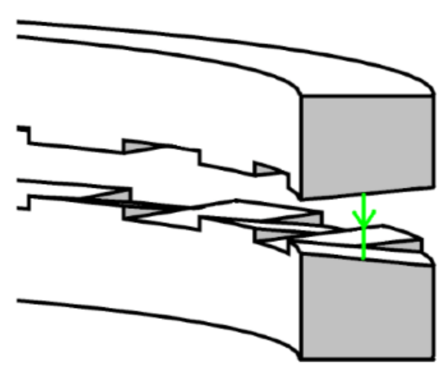
alternating gradient magnet (combined lens) system can provide strong focusing force in both directions



Even orbit is fixed, beam size becomes large as well as magnets.



Old Style
"Weak" Focus

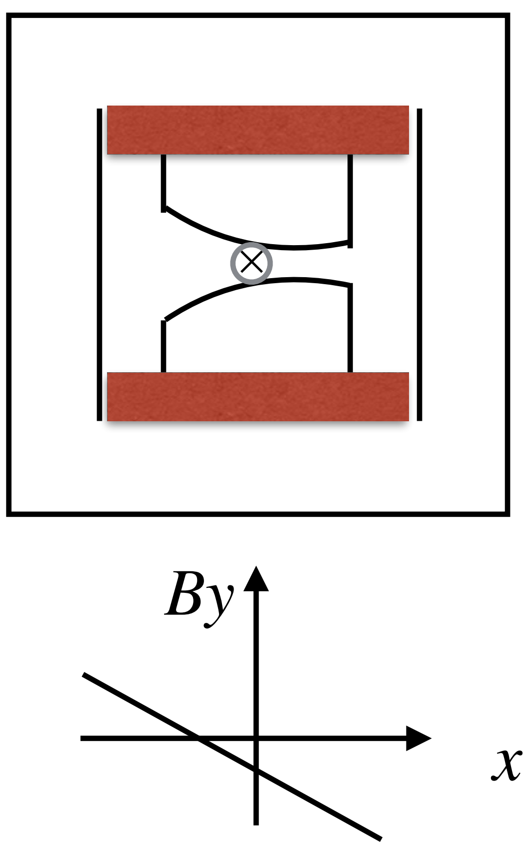


Alternating Gradient Synchrotron
Strong Focus

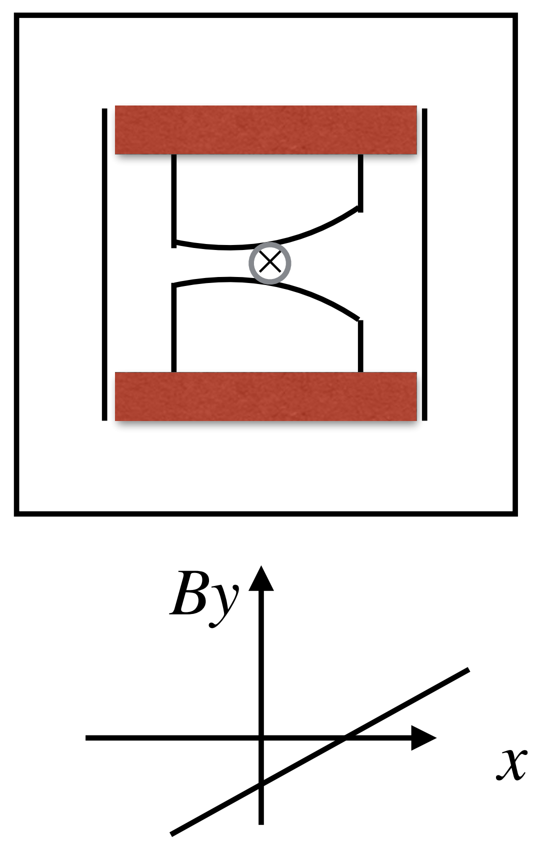
Combined function magnet

Bend + focus / defocus

BM + QF

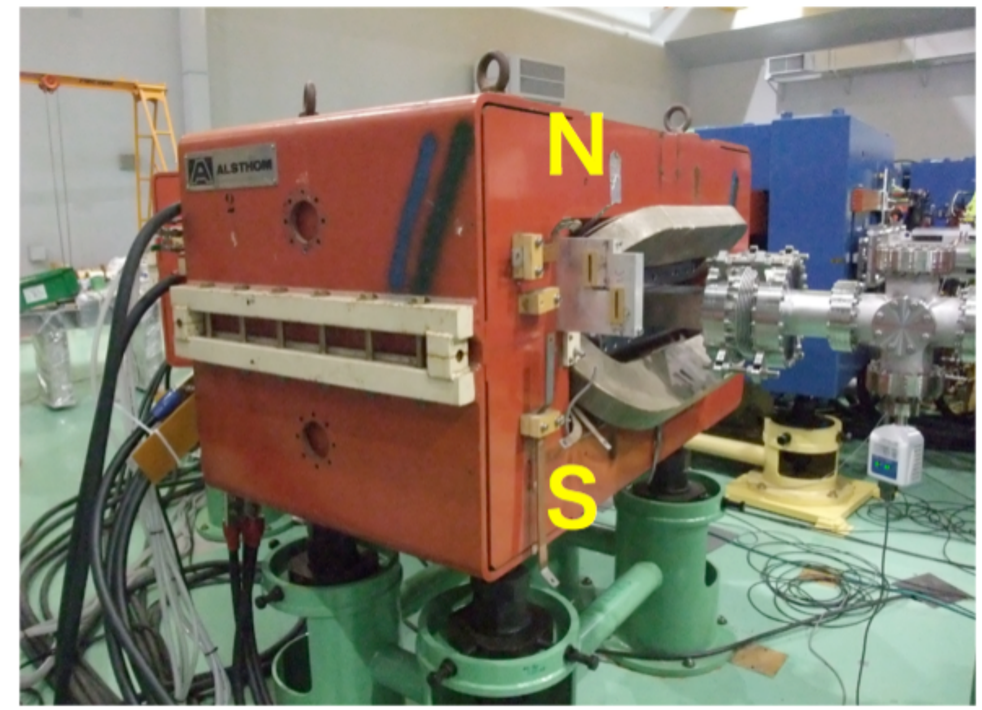


BM + QD

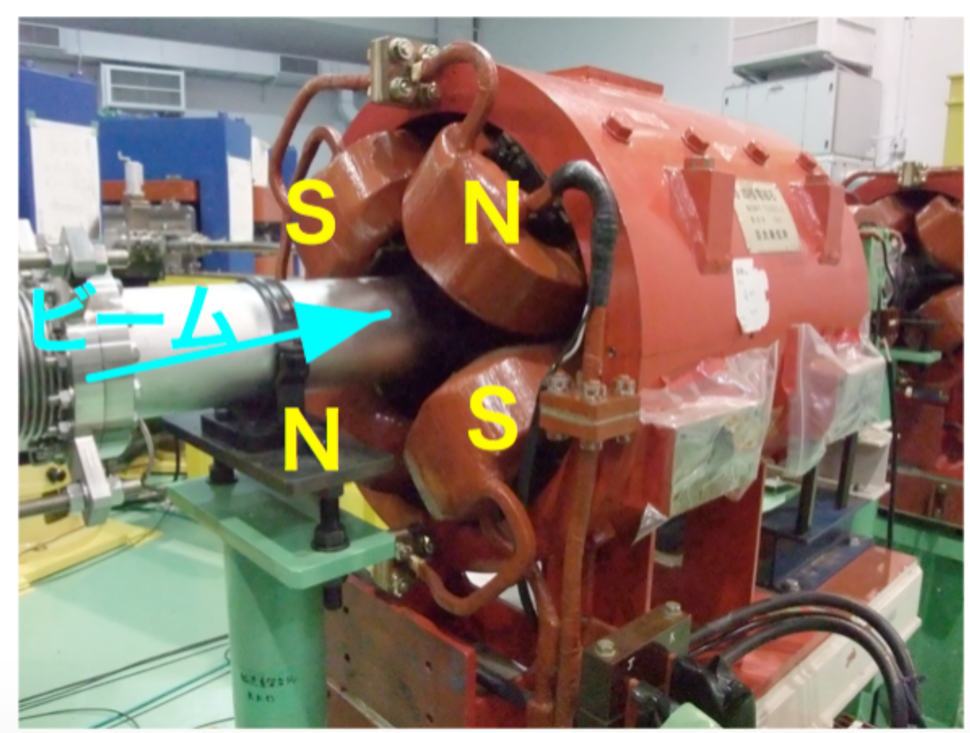


Separate function magnet

偏向電磁石（BM）



四極電磁石（QM）



piecewise solution (transfer matrix)

$$K(s) = 0 \qquad \begin{bmatrix} x \\ x' \end{bmatrix}_{\text{out}} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{\text{in}}$$

$$K(s) > 0 \qquad \begin{bmatrix} x \\ x' \end{bmatrix}_{\text{out}} = \begin{bmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{\text{in}}$$

$$K(s) < 0 \qquad \begin{bmatrix} x \\ x' \end{bmatrix}_{\text{out}} = \begin{bmatrix} \cosh(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sinh(\sqrt{K}L) \\ \sqrt{K}\sinh(\sqrt{K}L) & \cosh(\sqrt{K}L) \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{\text{in}}$$

closed form solution

Hill's equation

$$x'' + K(s)x = 0$$

K(s) has a periodicity

$$K(s+C) = K(s)$$

Then, general solution is given by the form of

$$x(s) = A\sqrt{\beta(s)}\cos[\psi(s) + \delta]$$

$$\frac{1}{2}\beta'' + K\beta - \frac{1}{\beta}\left[1 + \left(\frac{\beta'}{2}\right)^2\right] = 0$$

$$\alpha(s) = -\frac{1}{2}\frac{d\beta}{ds}$$

$$\gamma(s) = \frac{1+\alpha^2}{\beta}$$

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{s_0+C} = \begin{bmatrix} \cos\Delta\psi_c + \alpha\sin\Delta\psi_c & \beta\sin\Delta\psi_c \\ -\gamma\sin\Delta\psi_c & \cos\Delta\psi_c - \alpha\sin\Delta\psi_c \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{\mathrm{s}_0}$$

$$M(s_2|s_1) = \begin{bmatrix} \sqrt{\dfrac{\beta_2}{\beta_1}}(\cos\Delta\psi + \alpha_1\sin\Delta\psi) & \sqrt{\beta_1\beta_2}\sin\Delta\psi \\ -\dfrac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\Delta\psi + \dfrac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\Delta\psi & \sqrt{\dfrac{\beta_1}{\beta_2}}(\cos\Delta\psi - \alpha_2\sin\Delta\psi) \end{bmatrix}$$

$$M(s_2|s_1) = \begin{bmatrix} \sqrt{\beta_2} & 0 \\ -\dfrac{\alpha_2}{\sqrt{\beta_2}} & \dfrac{1}{\sqrt{\beta_2}} \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \dfrac{1}{\sqrt{\beta_1}} & 0 \\ \dfrac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{bmatrix}$$

$$\psi(s) = \int_0^s \frac{ds'}{\beta(s')} \qquad\qquad \nu = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

phase advance — tune

Action-angle variable and Courant Snyder invariant

Action angle variable in betatron oscillations

We start from linearized Hamiltonian to describe the betatron motion with action-angle variables.

$$H_{\mathrm{lin}} = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}h^2x^2 + \frac{B_1}{2B\rho}(x^2 - y^2)$$

We are looking at only in x direction.

harmonic oscillation

$$H(x,p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2$$

$$\begin{aligned} x &= a\cos(\omega t + \phi_0) \\ p &= -a\omega\sin(\omega t + \phi_0) \end{aligned}$$

betatron oscillation

$$\begin{aligned} H &= \frac{1}{2}p^2 + \frac{1}{2}\left(h^2(s) + \frac{B_1(s)}{B\rho}\right)x^2 \\ &= \frac{1}{2}p^2 + \frac{1}{2}K(s)x^2 \end{aligned}$$

$$\begin{aligned} x &= a\sqrt{\beta}\cos(\nu\phi(s) + \delta) \\ p &= -a\frac{1}{\sqrt{\beta}}\left[\sin(\nu\phi(s) + \delta) + \alpha\cos(\nu\phi(s) + \delta)\right] \end{aligned}$$

Introduce a new set of canonical variables (ψ, J).

harmonic oscillation

$$\begin{aligned} x &= a(J)\cos\phi \\ p &= -a(J)\omega\sin\phi \end{aligned}$$

$$p = -\omega x \tan\phi$$

$$F_1(x,\phi) = -\frac{\omega x^2}{2}\tan\phi$$

$$\begin{aligned} J &= -\frac{\partial F_1}{\partial \phi} = \frac{\omega x^2}{2}\frac{1}{\cos^2\phi} \\ &= \frac{\omega x^2}{2}(1+\tan^2\phi) \\ &= \frac{\omega x^2}{2}\left(1+\frac{p^2}{\omega^2 x^2}\right) \\ &= \frac{1}{2\omega}(\omega^2 x^2 + p^2) \end{aligned}$$

betatron oscillation

$$\begin{aligned} x &= a(J)\sqrt{\beta}\cos(\psi) \\ p &= -a(J)\frac{1}{\sqrt{\beta}}(\sin\psi + \alpha\cos\psi) \end{aligned}$$

$$p = -\frac{x}{\beta}(\tan\psi + \alpha)$$

$$F_1(x,\psi) = -\frac{x^2}{2\beta}(\tan\psi + \alpha)$$

$$\begin{aligned} J &= -\frac{\partial F_1}{\partial \psi} = \frac{x^2}{2\beta}\frac{1}{\cos^2\psi} \\ &= \frac{x^2}{2\beta}(1+\tan^2\psi) \\ &= \frac{1}{2\beta}[x^2 + (\beta p + \alpha x)^2] \end{aligned}$$

harmonic oscillation

$$x = \sqrt{2J/\omega}\cos\phi$$

$$p = -\sqrt{2J\omega}\sin\phi$$

$$\begin{aligned} H_1 &= H(\phi, J) + \frac{\partial F_1}{\partial t} \\ &= J\omega \end{aligned}$$

$$\frac{1}{2}\beta'' + K\beta - \frac{1}{\beta}\left[1 + \left(\frac{\beta'}{2}\right)^2\right] = 0 \quad \text{or}$$

$$\alpha' = K\beta - \frac{1}{\beta}(1+\alpha^2)$$

betatron oscillation

$$x = \sqrt{2\beta J}\cos\psi$$

$$p = -\sqrt{\frac{2J}{\beta}}[\sin\psi + \alpha\cos\psi]$$

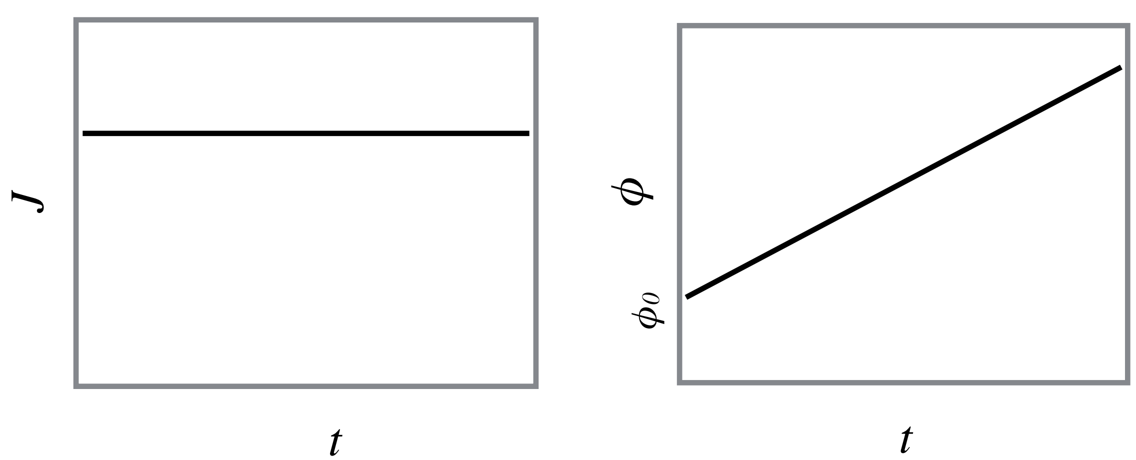
$$\begin{aligned} H_1 &= H + \frac{\partial F_1}{\partial s} \\ &= \frac{1}{2}p^2 + \frac{1}{2}K(s)x^2 + \frac{\partial}{\partial s}\frac{x^2}{2\beta}\left(\frac{\beta'}{2} - \tan\psi\right) \\ &= \frac{1}{2}p^2 + \frac{1}{2}K(s)x^2 + \frac{x^2}{4}\frac{\beta'' - \beta'^2}{\beta^2} + \frac{x^2\beta'}{2\beta^2}\tan\psi \end{aligned}$$

$$\begin{aligned} H_1 &= \frac{1}{2}p^2 + \frac{1}{2}K(s)x^2 + \frac{x^2}{4}\frac{\beta'^2/2 + K\beta^2 - 2}{\beta^2} + \frac{x^2\beta'}{2\beta^2}\left(\frac{\beta p}{x} + \alpha\right) \\ &= \frac{1}{2}p^2 + \frac{1}{2\beta^2}x^2 + \frac{\alpha^2}{2\beta^2}x^2 + \frac{\alpha}{\beta}px \\ &= \frac{1}{2\beta^2}x^2 + \frac{1}{2}\left(p + \frac{\alpha}{\beta}x\right)^2 \\ &= \frac{J}{\beta} \end{aligned}$$

harmonic oscillation

$$
\begin{aligned}
\dot{J} &= -\frac{\partial H_1}{\partial \phi} = 0 \\
\dot{\phi} &= \frac{\partial H_1}{\partial J} = \omega
\end{aligned}
$$

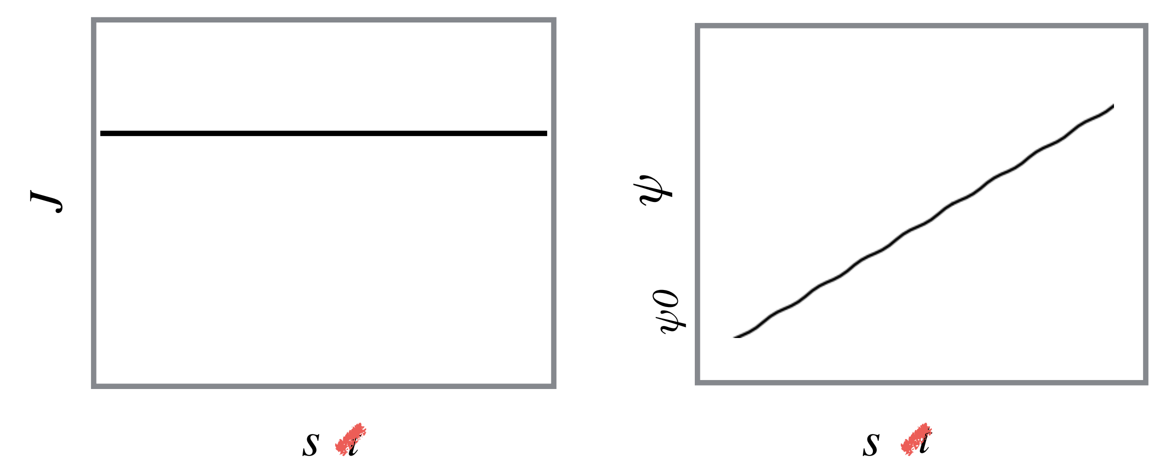
$$
\begin{aligned}
J &= \text{constant} \\
\phi &= \omega t + \phi_0
\end{aligned}
$$



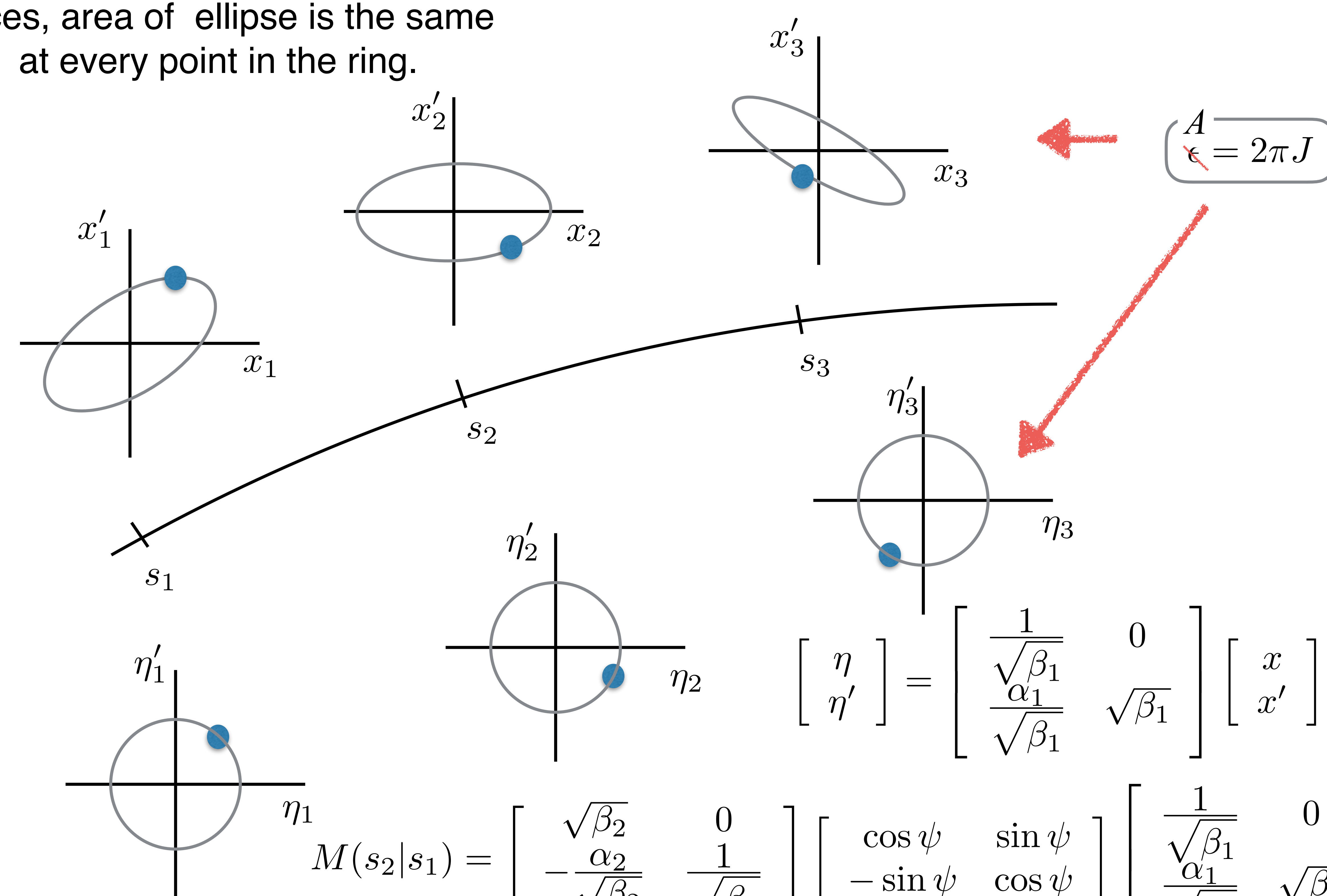
betatron oscillation

$$
\begin{aligned}
J' &= -\frac{\partial H_1}{\partial \psi} = 0 \\
\psi' &= \frac{\partial H_1}{\partial J} = \frac{1}{\beta(s)}
\end{aligned}
\tag{5A.21}
$$

$$
\begin{aligned}
J &= \text{constant} \\
\psi &= \int_0^s \frac{1}{\beta(s')} ds'
\end{aligned}
\tag{5A.22}
$$



If the motion is governed by linear forces, area of ellipse is the same at every point in the ring.



$$\begin{bmatrix} \eta \\ \eta' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ \frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}$$

$$M(s_2|s_1) = \begin{bmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ \frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{bmatrix}$$

$$2J = \gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

Courant Snyder invariant

Beta function

$\alpha(s_2)$, $\beta(s_2)$ can be expressed using $\alpha(s_1)$, $\beta(s_1)$.

$$\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_2 = \begin{bmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22}+M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_1$$

Why ? $2J$ is constant in the ring.

$$\begin{aligned} 2J &= \gamma_1 x_1^2 + 2\alpha_1 x_1 x_1' + \beta_1 x_1'^2 \\ &= \gamma_2 x_2^2 + 2\alpha_2 x_2 x_2' + \beta_2 x_2'^2 \end{aligned}$$

using transfer matrix

$$\begin{aligned} x_2 &= M_{11}x_1 + M_{12}x_1' \\ x_2' &= M_{21}x_1 + M_{22}x_1' \end{aligned}$$

solve for x_1 and x'_1

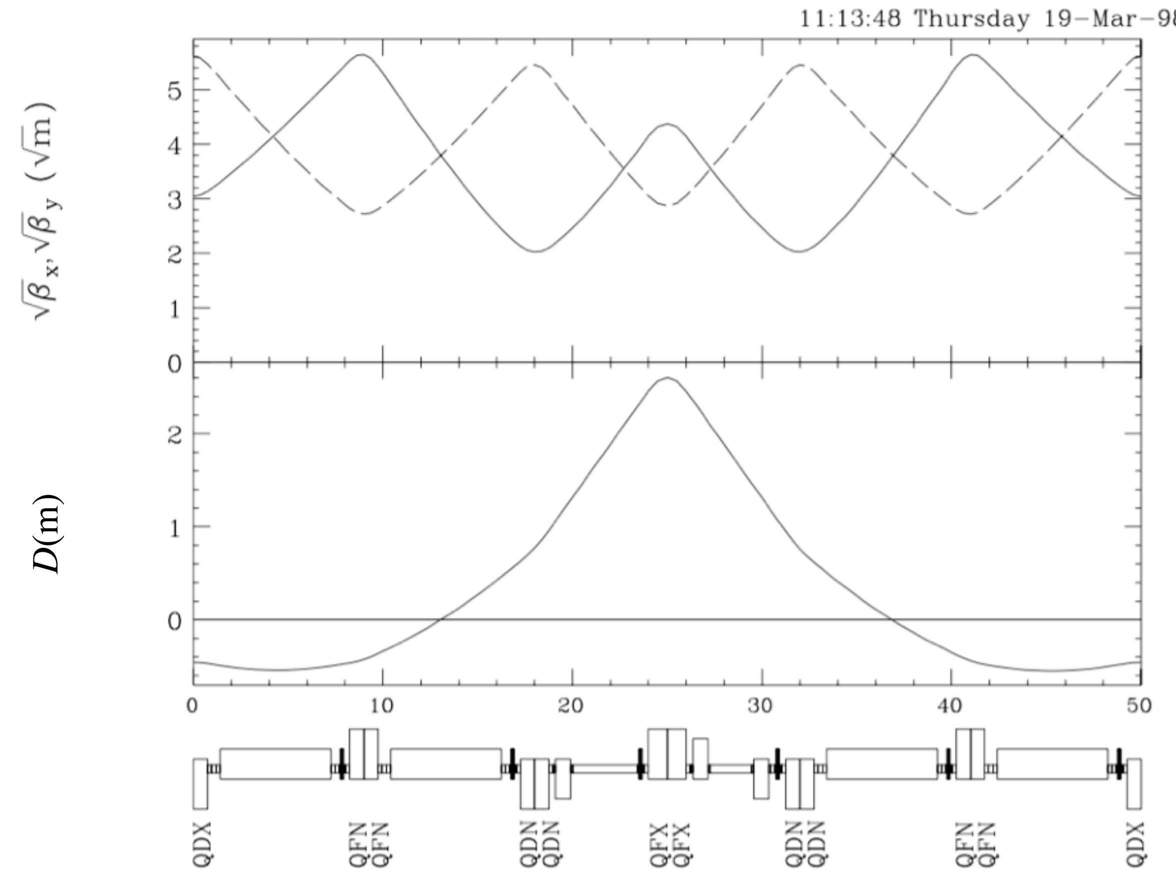
$$\begin{aligned} x_1 &= M_{22}x_2 - M_{12}x_2' \\ x_1' &= -M_{21}x_2 + M_{22}x_2' \end{aligned}$$

$$
M(s_2|s_1) = \begin{bmatrix} \cos\Delta\psi_c + \alpha\sin\Delta\psi_c & \beta\sin\Delta\psi_c \\ -\gamma\sin\Delta\psi_c & \cos\Delta\psi_c - \alpha\sin\Delta\psi_c \end{bmatrix}
$$

$$
\Delta\psi_c = \arccos\frac{M_{11}+M_{22}}{2}
$$

$$
\beta = \frac{M_{12}}{\sin\Delta_c}
$$

$$
\gamma = -\frac{M_{21}}{\sin\Delta_c}
$$



How to draw Beta function

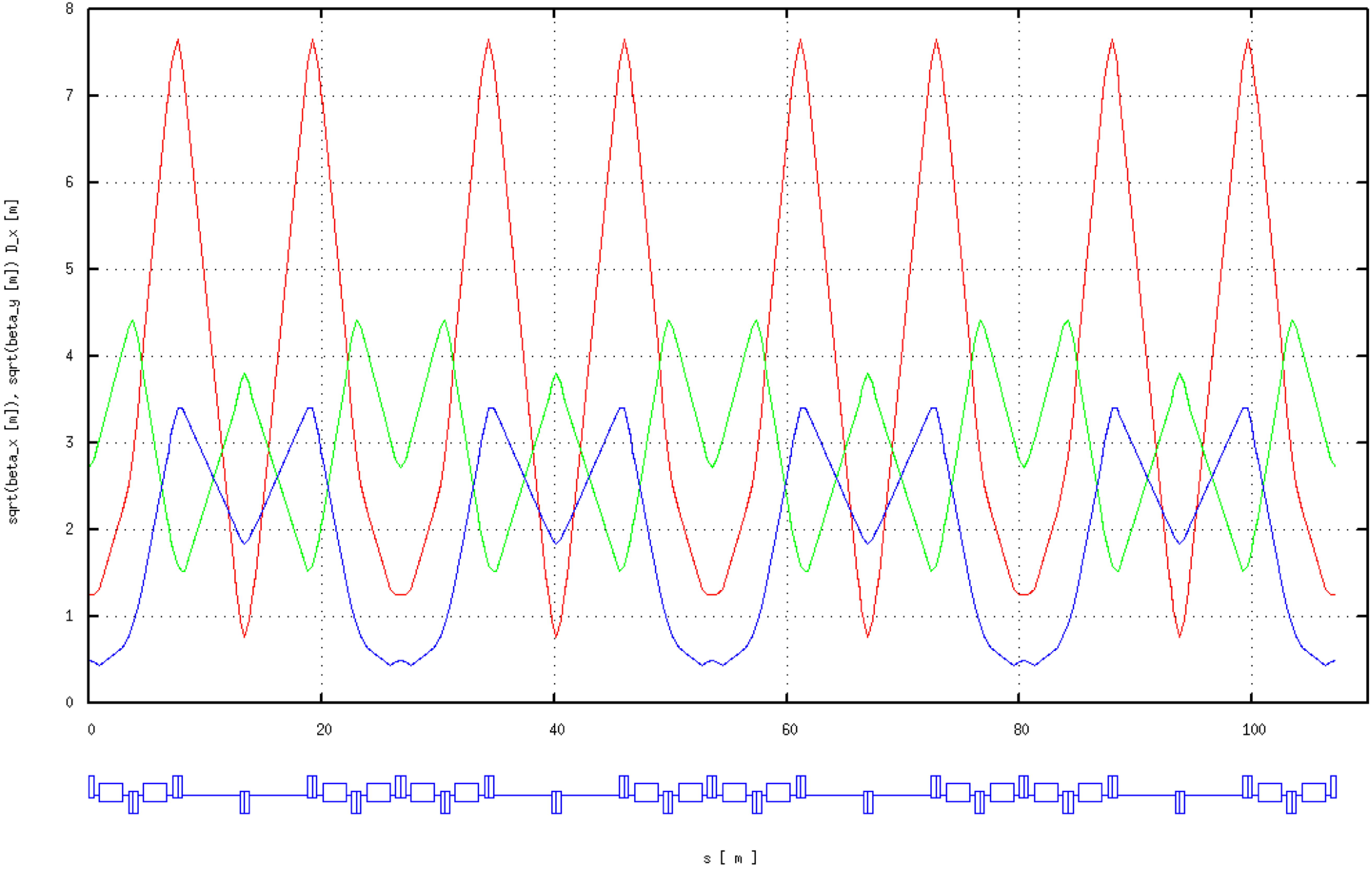
Check spec of every element in 1 period.

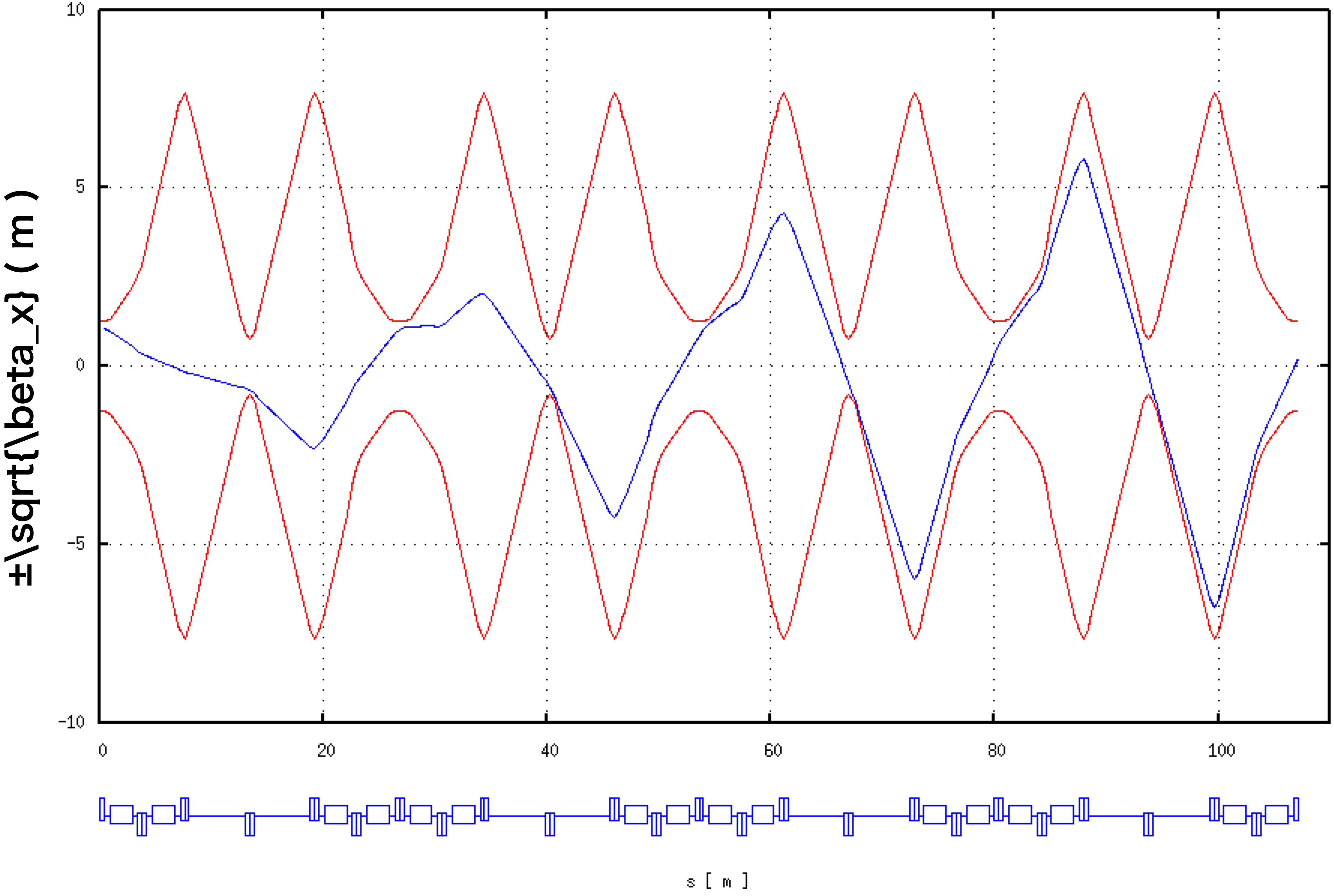
Concatenate transfer matrix for 1 period.

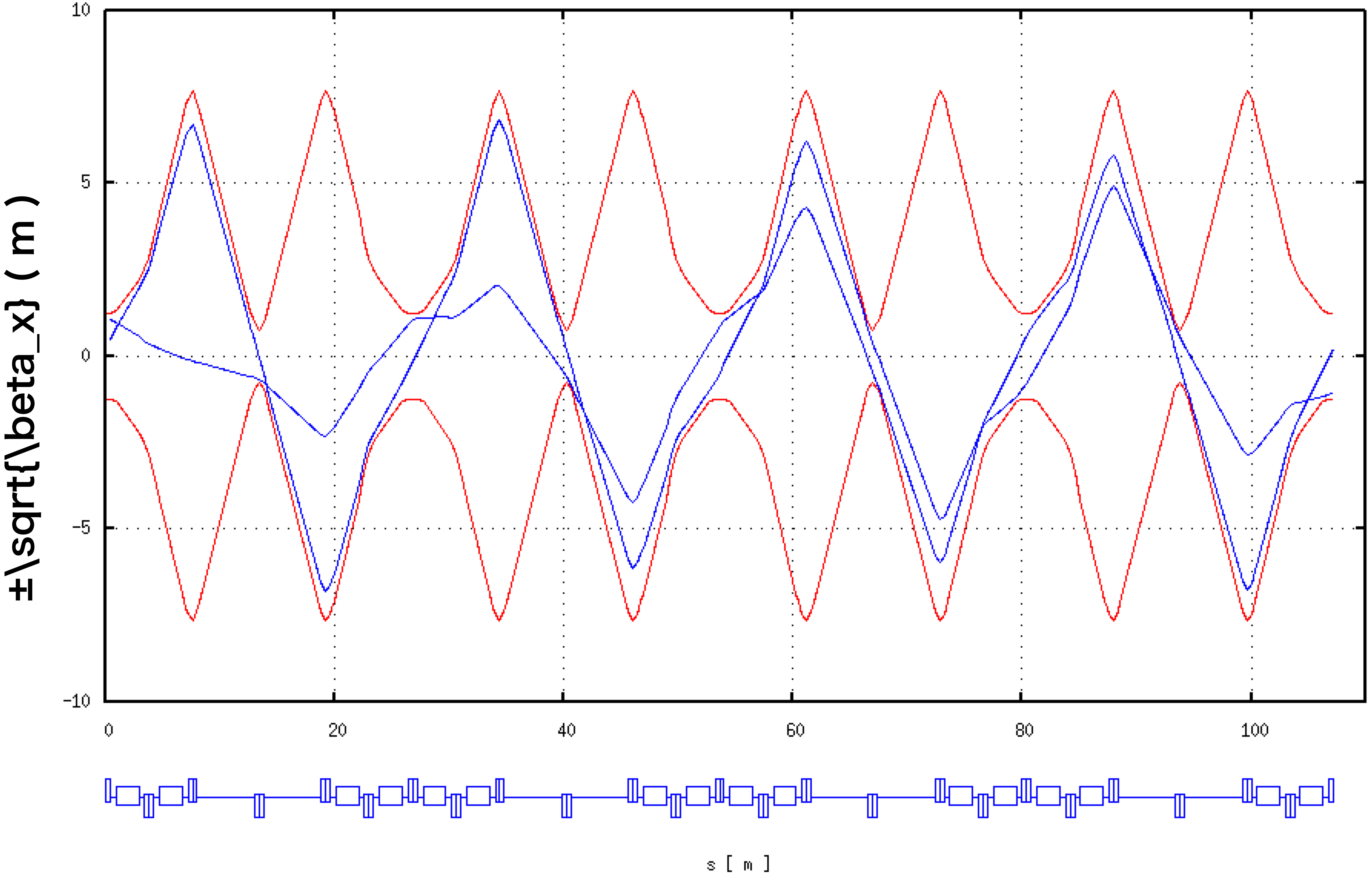
α, β, γ at the start are determined.

$$
\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_2 = \begin{bmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22}+M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_1
$$

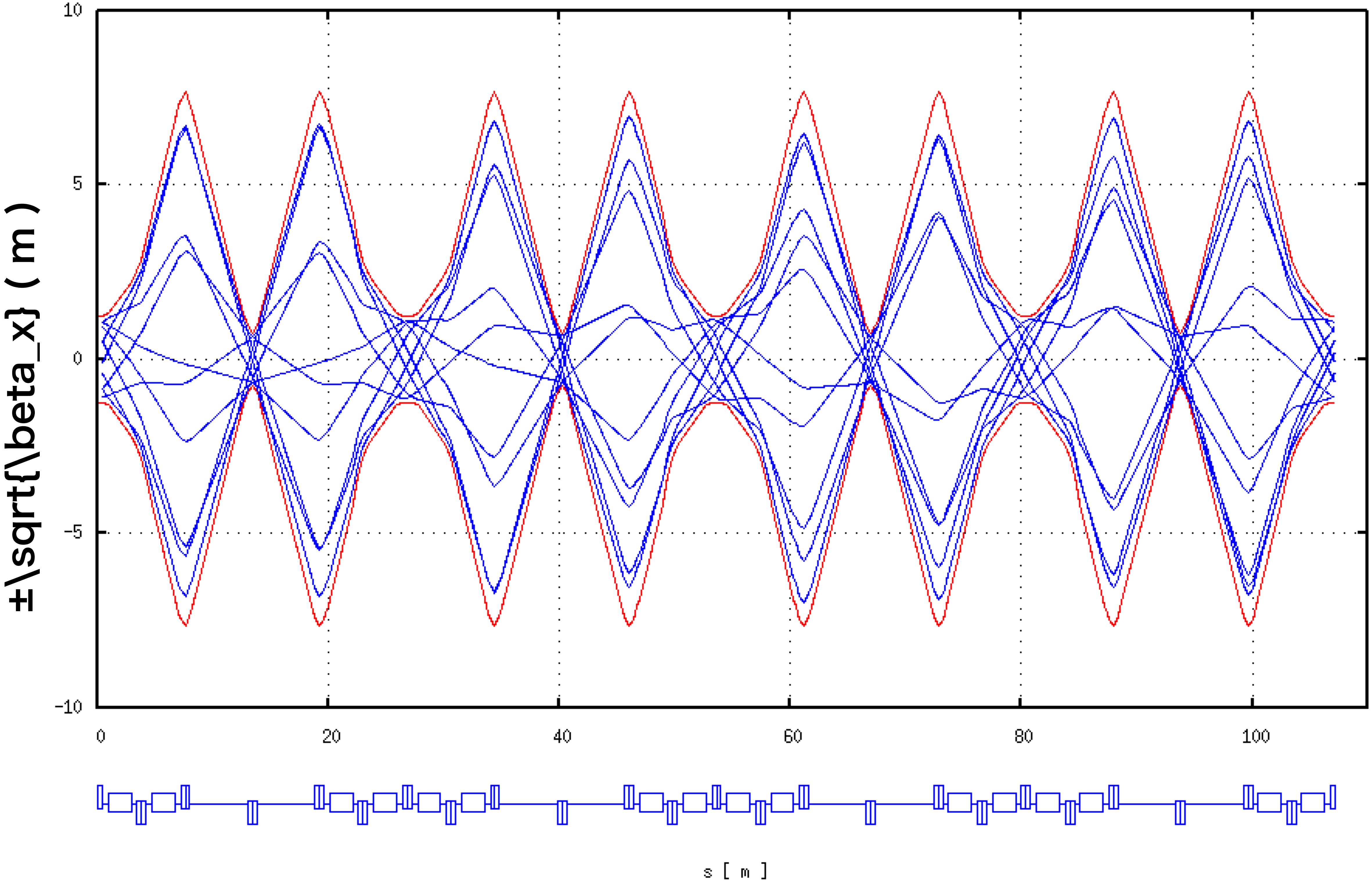
Then, use the matrix in left to transform α,β,γ.

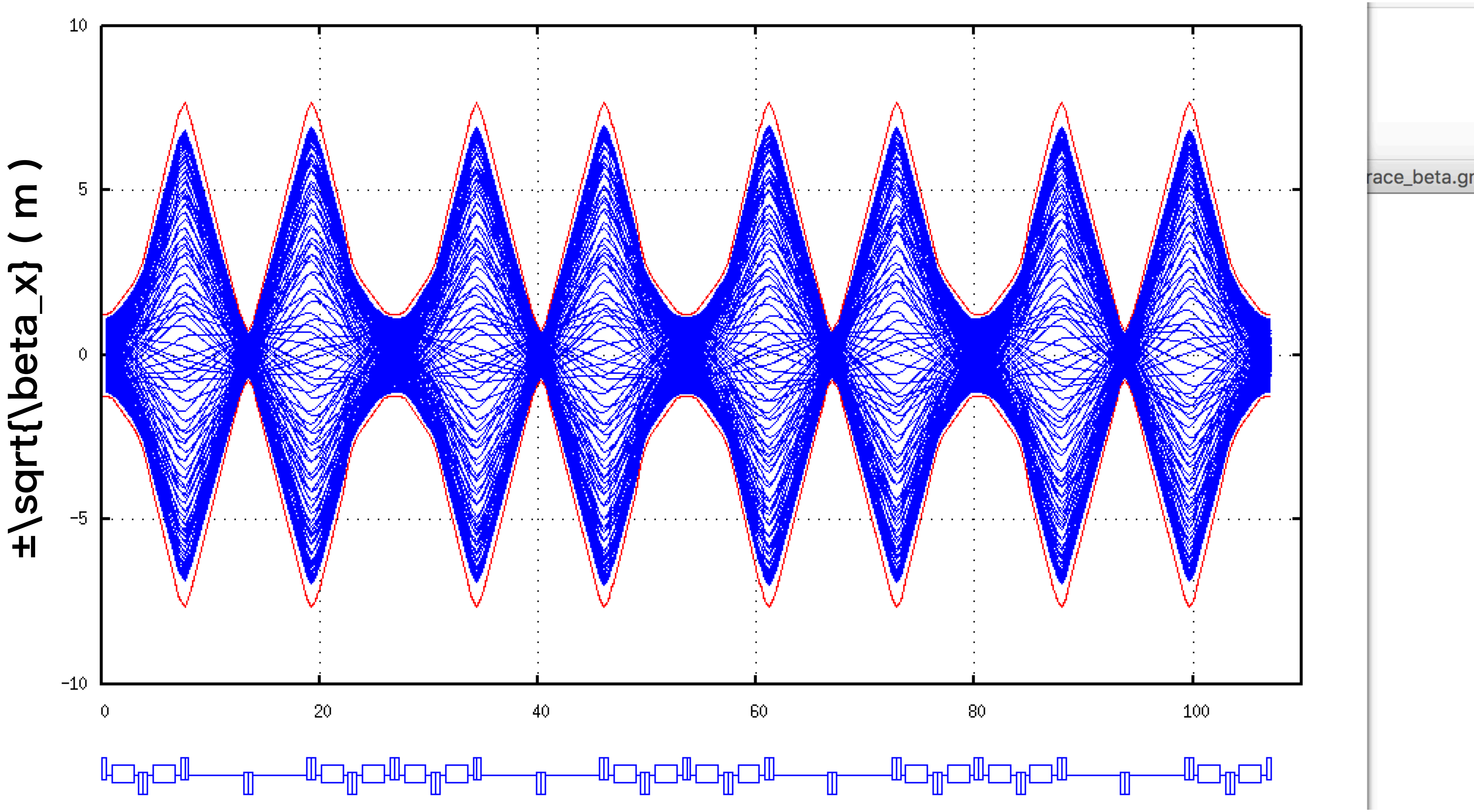

sqrt(beta_x [m]), sqrt(beta_y [m]) D_x [m]
0
1
2
3
4
5
6
7
8
0
20
40
60
80
100
s [m]

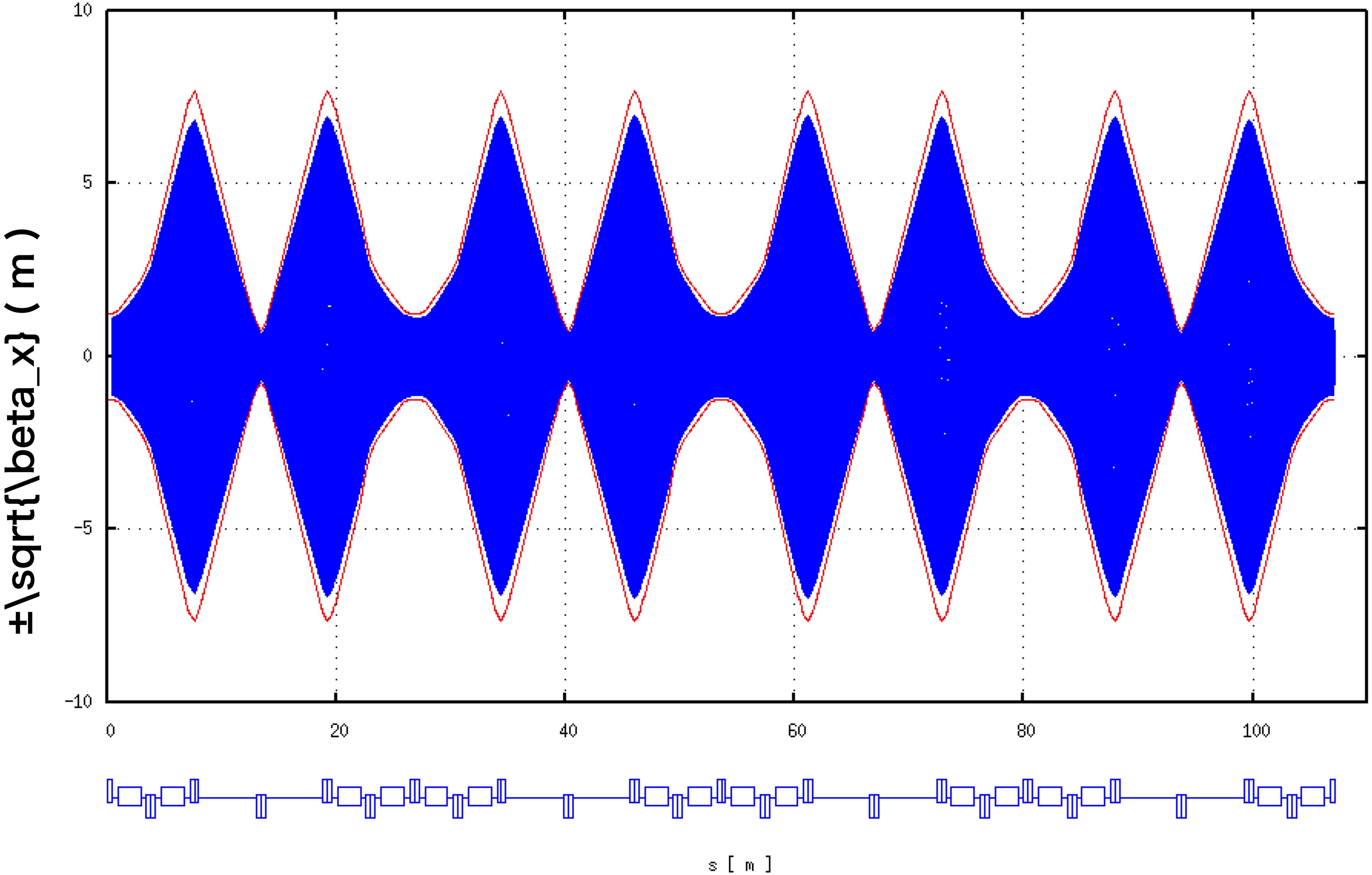

±\sqrt{\beta_x} (m)
10
5
0
-5
-10
0
20
40
60
80
100
s [m]



±\sqrt{\beta_x} (m)
10
5
0
-5
-10
0
20
40
60
80
100
s [m]

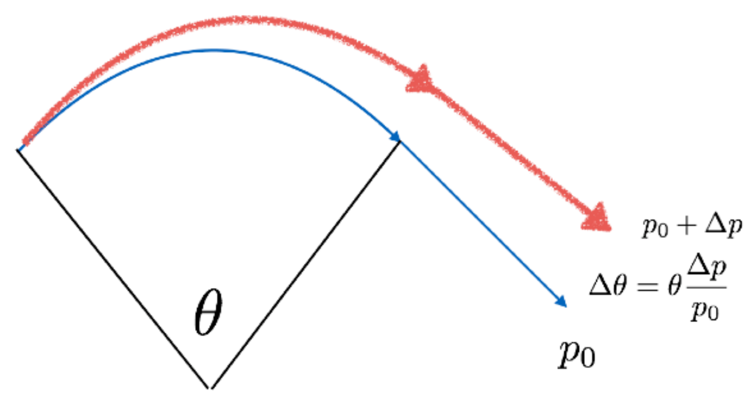

±\sqrt{\beta_x} (m)
10
5
0
-5
-10
0
20
40
60
80
100
s [m]


±\sqrt{\beta_x} (m)
10
5
0
-5
-10
0
20
40
60
80
100
race_beta.gr


±\sqrt{\beta_x} (m)
10
5
0
-5
-10
0
20
40
60
80
100
s [m]

Off momentum orbit and dispersion function

Off-momentum particle



Particle with higher momentum has a larger radius of curvature, because it is hard to bend.

$$\mathcal{H} \sim \frac{\mathcal{P}_x^2 + \mathcal{P}_y^2}{2(1+\delta)} - hx\delta + \frac{h^2x^2}{2} + \frac{K_1}{2}(x^2 - y^2)$$

$$x'' + K_x(s)x = \frac{\delta}{\rho}$$

$$x(s) = x_\beta(s) + D(s)\delta$$

$$\begin{aligned} x_\beta''(s) + K_x(s)x_\beta(s) &= 0 \\ D''(s) + K_x(s)D(s) &= \frac{1}{\rho} \end{aligned}$$

D(s): Dispersion function

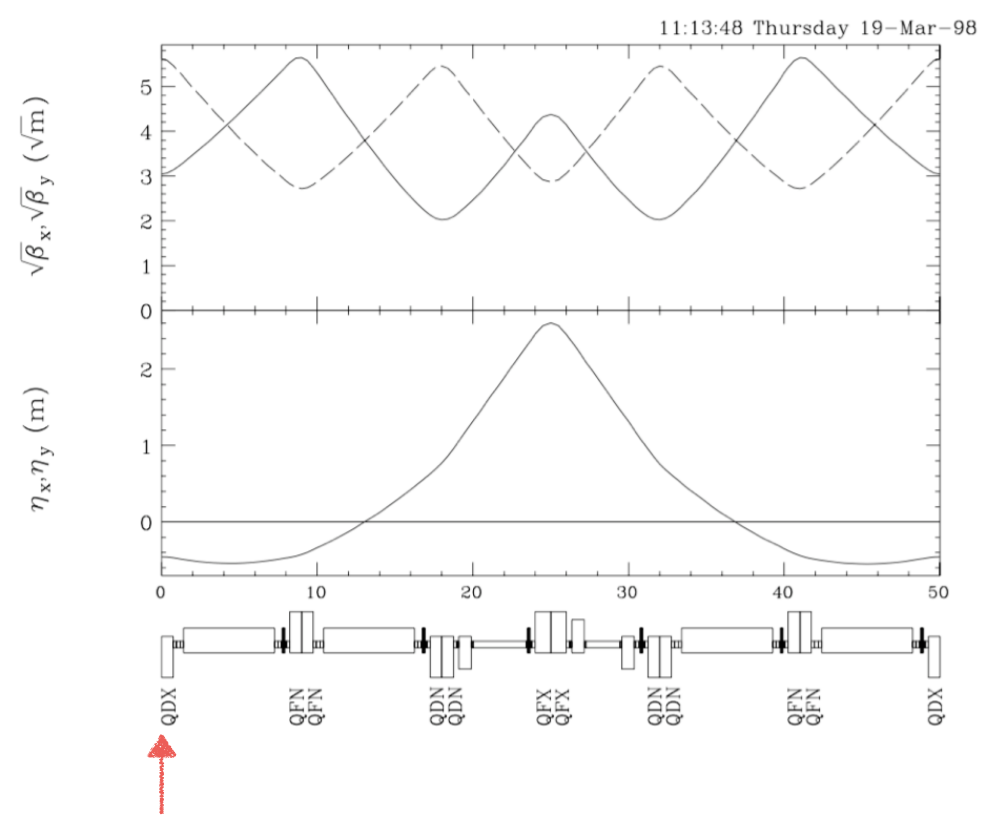
$$\Delta x(s) = D(s)\delta$$
$$\Delta x'(s) = D'(s)\delta$$

$$\begin{bmatrix} D \\ D' \end{bmatrix}_{\text{out}} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} D \\ D' \end{bmatrix}_{\text{in}} + \begin{bmatrix} d \\ d' \end{bmatrix}$$

K	d	d'
negative	$\frac{e}{p\lvert K\rvert} B_0 (\cosh \sqrt{\lvert K\rvert} L - 1)$	$\frac{e}{p\sqrt{\lvert K\rvert}} B_0 \sinh \sqrt{\lvert K\rvert} L$
0	$\frac{1}{2} \frac{B_0 L}{p} L$	$\frac{B_0 L}{p}$
positive	$\frac{e}{pK} B_0 (1 - \cos \sqrt{K} L)$	$\frac{e}{p\sqrt{K}} B_0 \sin \sqrt{\lvert K\rvert} L$

$$\begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{\text{out}} = \begin{bmatrix} M_{11} & M_{12} & d \\ M_{21} & M_{22} & d' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{\text{in}}$$

$$\begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{\text{in}} = \boldsymbol{M}_{\text{periodic}} \begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{\text{in}}$$



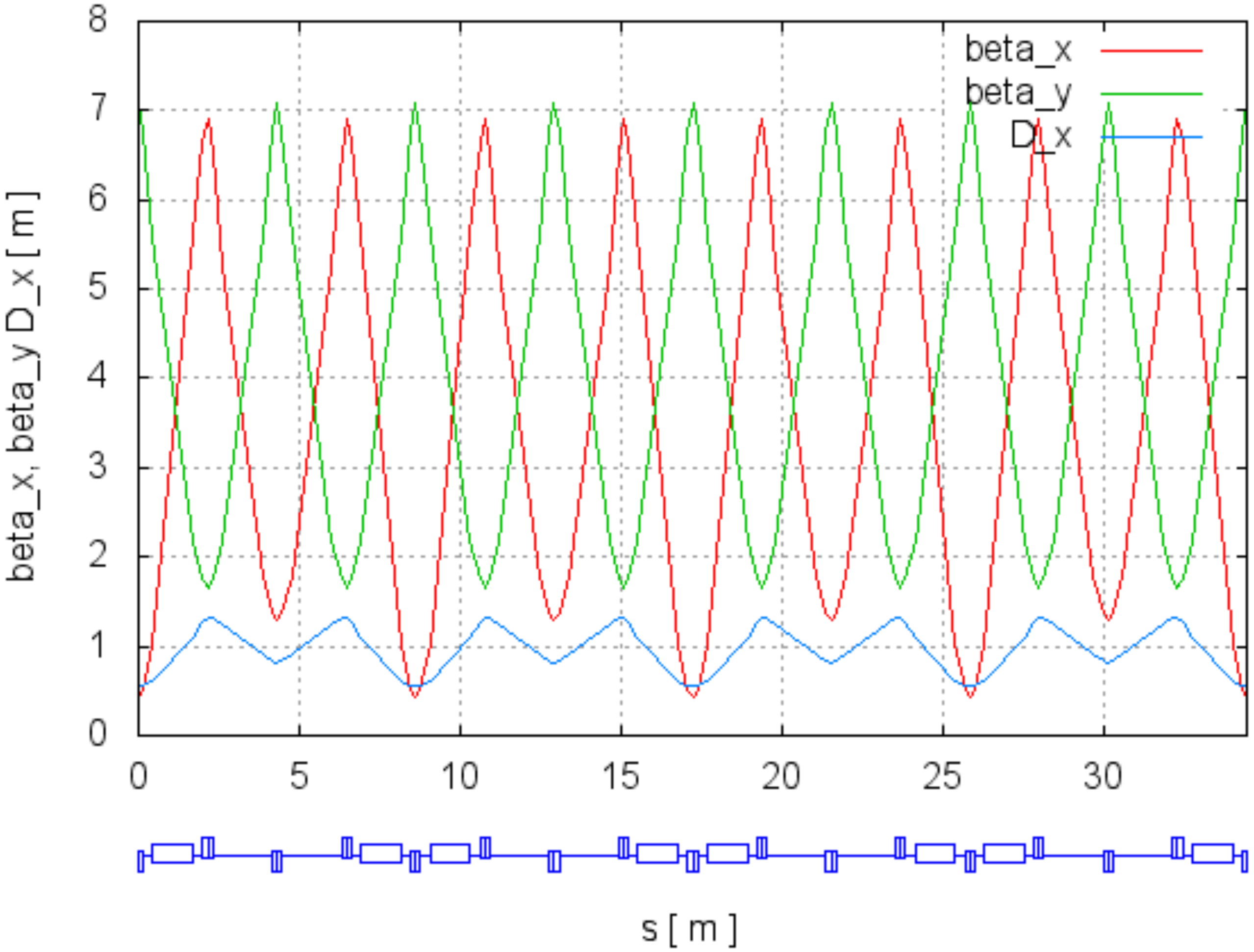
$$\begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{\text{in}}$$

Initial values

$$D = \frac{M_{13}(1 - M_{22}) + M_{12} M_{23}}{2 - M_{11} - M_{22}} = \frac{M_{13}(1 - \cos\Phi + \alpha \sin\Phi) + M_{23} \beta \sin\Phi}{2(1 - \cos\Phi)}$$

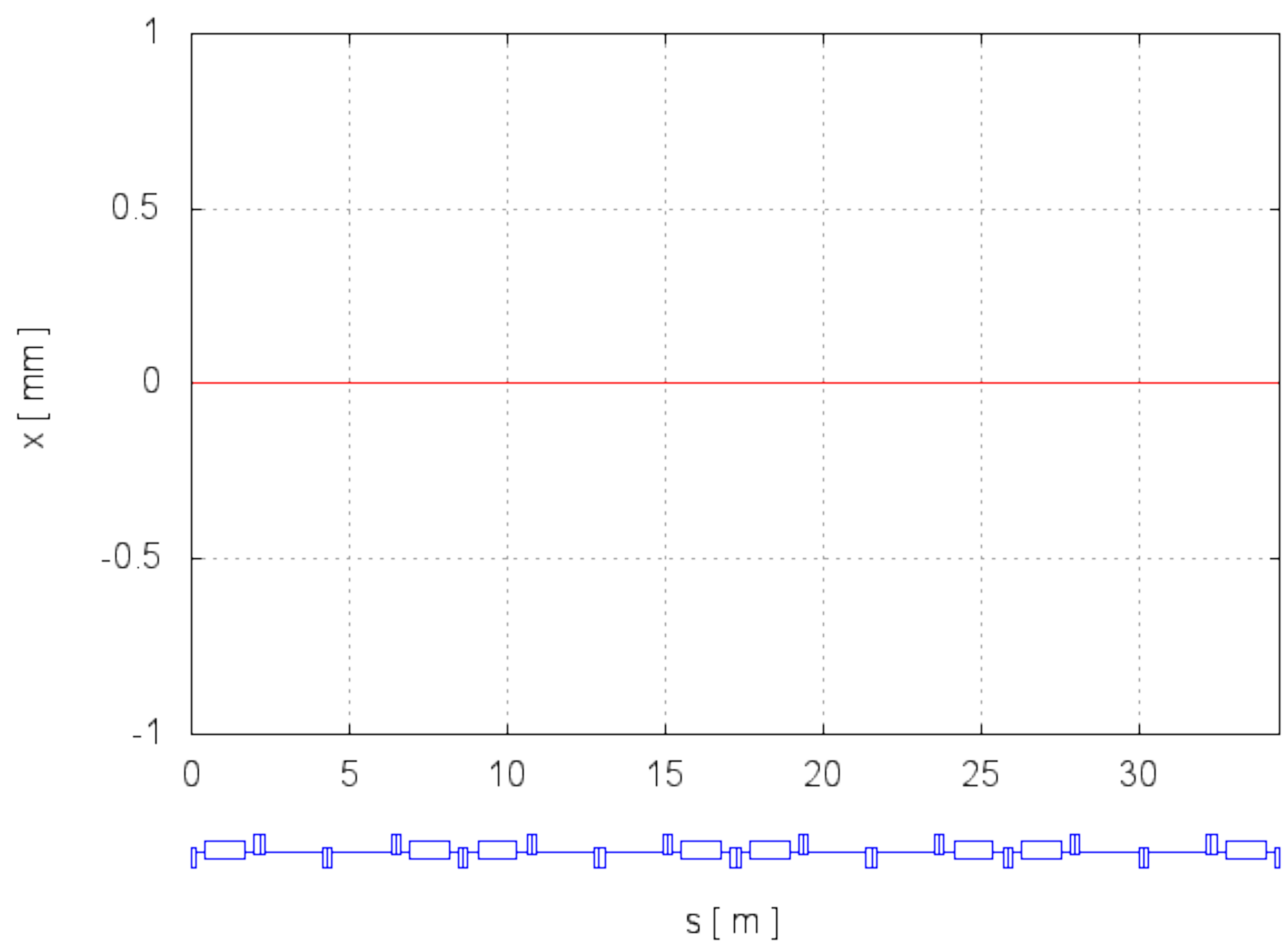
$$D' = \frac{M_{13} M_{21} + M_{23}(1 - M_{11})}{2 - M_{11} - M_{22}} = \frac{-M_{13} \gamma \sin\Phi + M_{23}(1 - \cos\Phi - \alpha \sin\Phi)}{2(1 - \cos\Phi)}$$

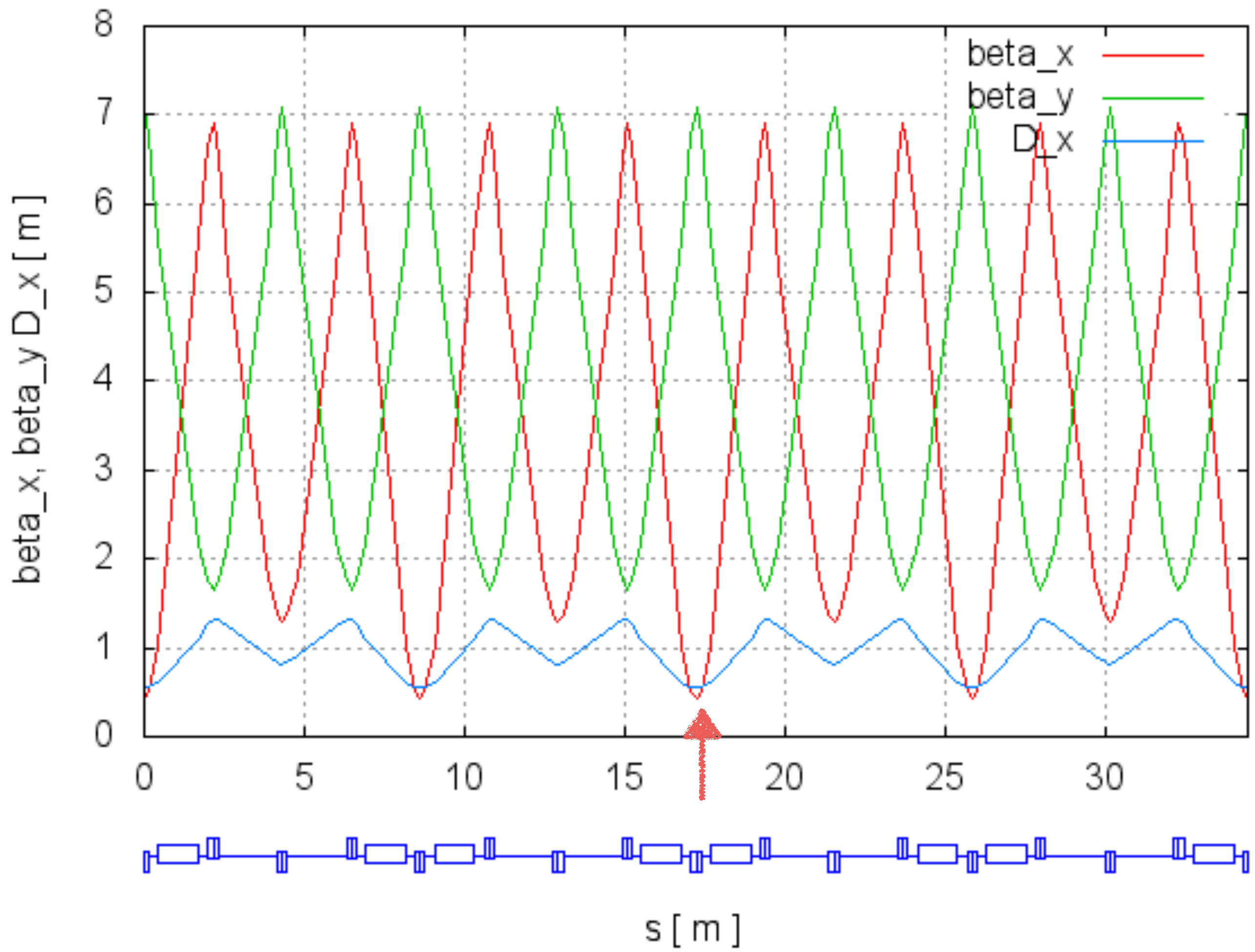
Dipole error and COD (Closed Orbit Distortion)



w /o dipole error

Launch the beam with (x,x') = (0.0 mm, 0.0mr). Of course it goes straight with no amplitude.

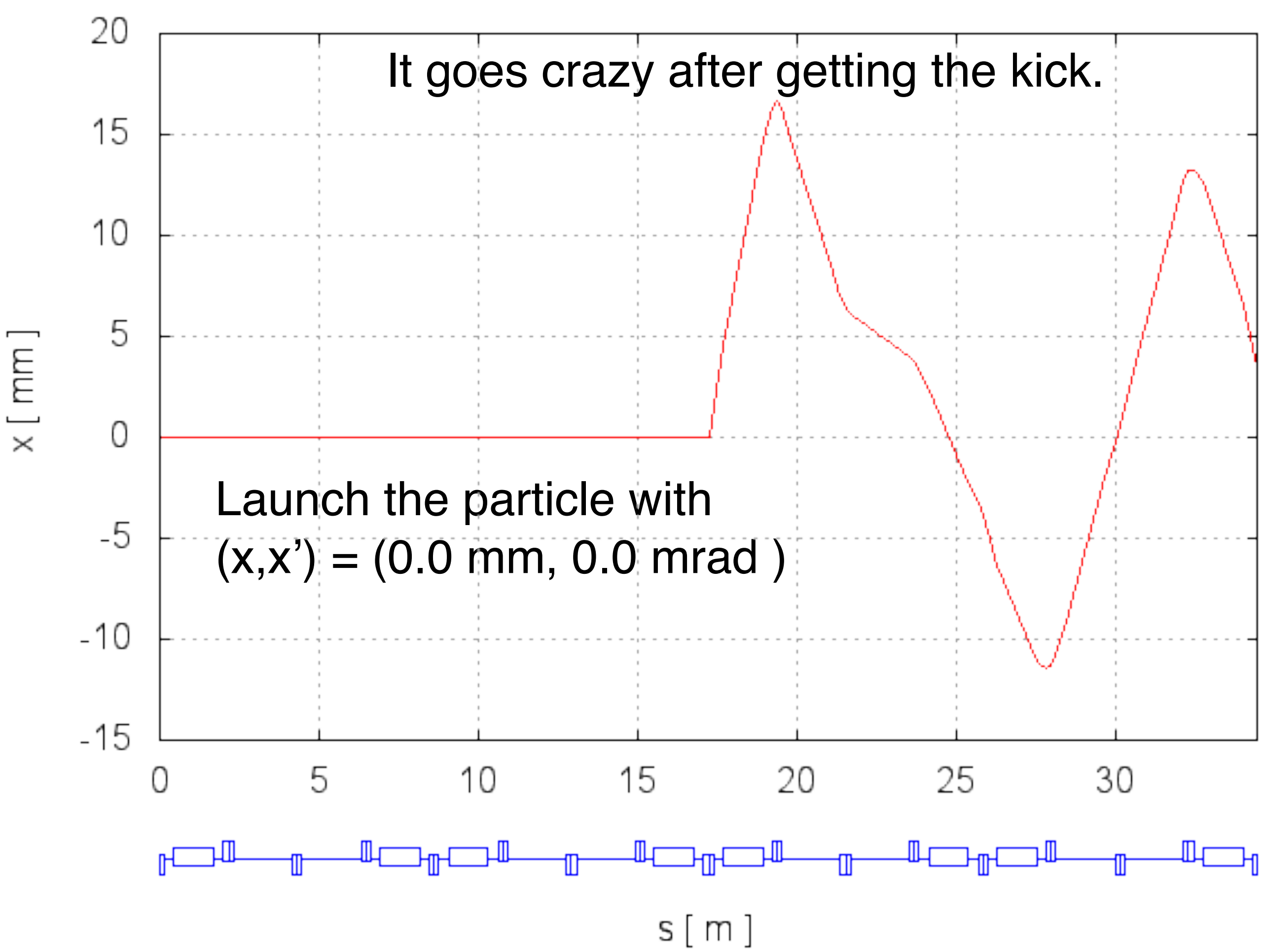




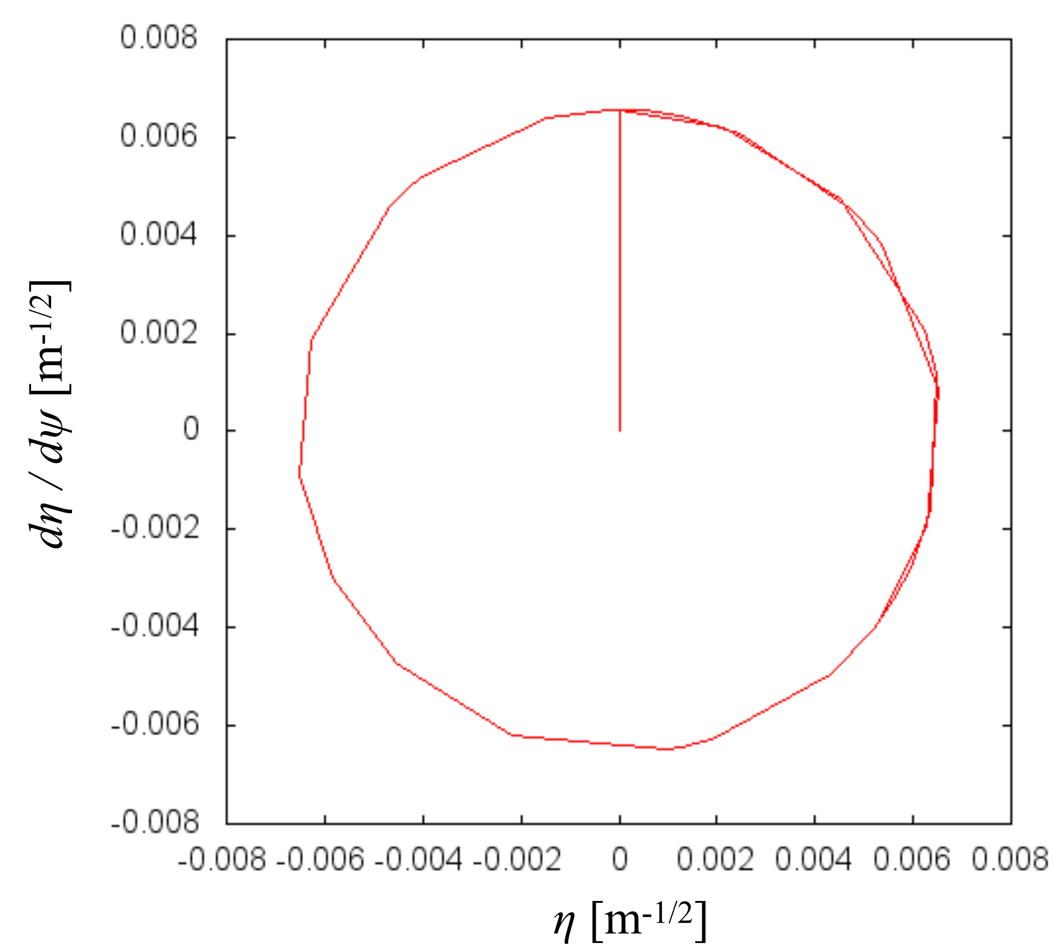
Suppose there is a dipole error of 10 mad at QD.

$$\eta = x/\sqrt{\beta}$$

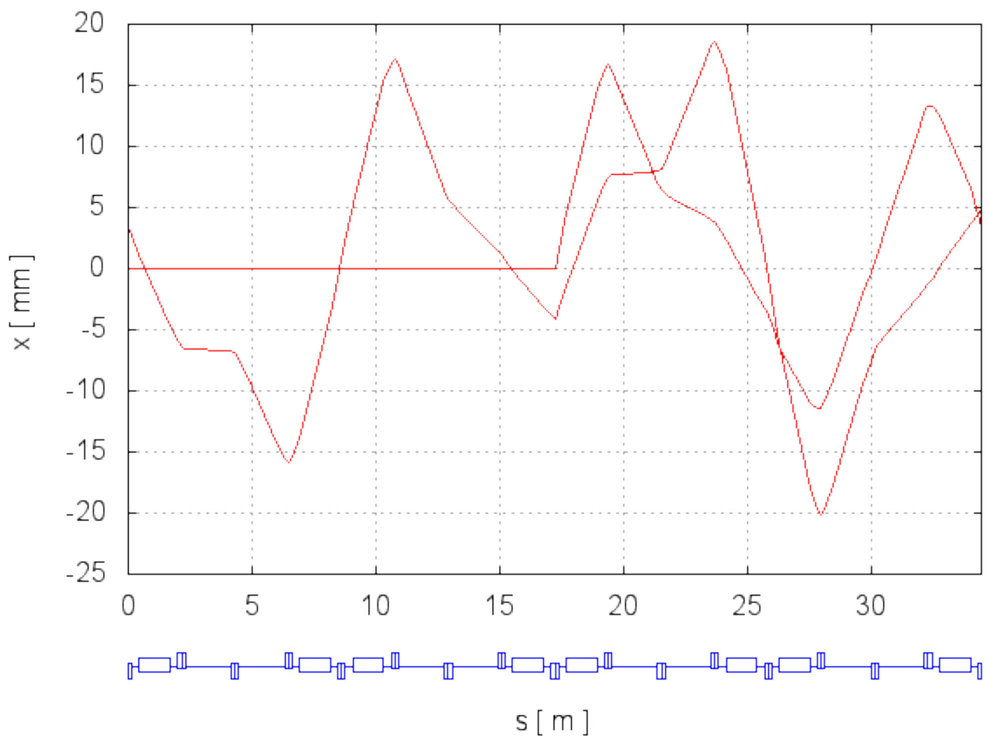
$$d\eta/d\psi = x\alpha/\sqrt{\beta} + x'\sqrt{\beta}$$



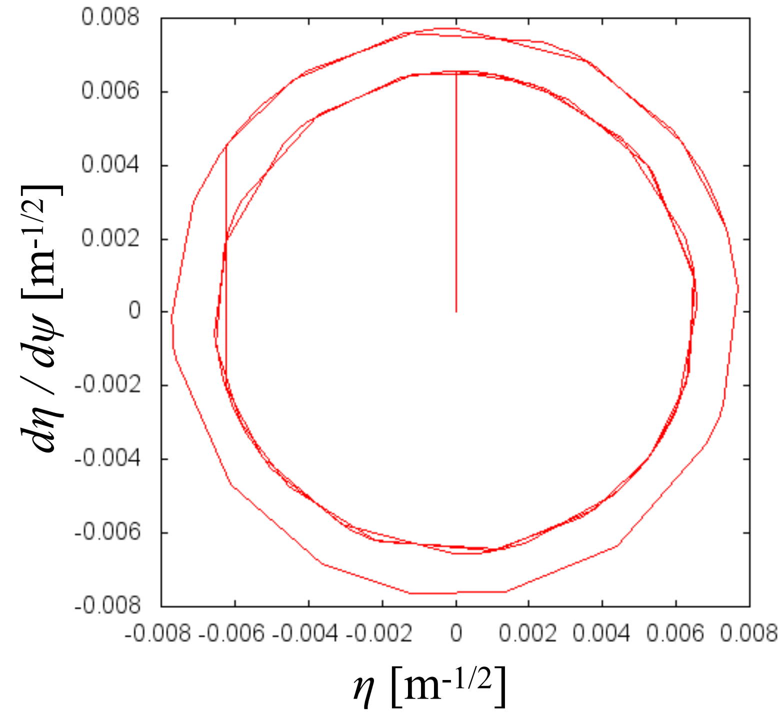
Trajectory at every element of the particle in the normalized phase space.



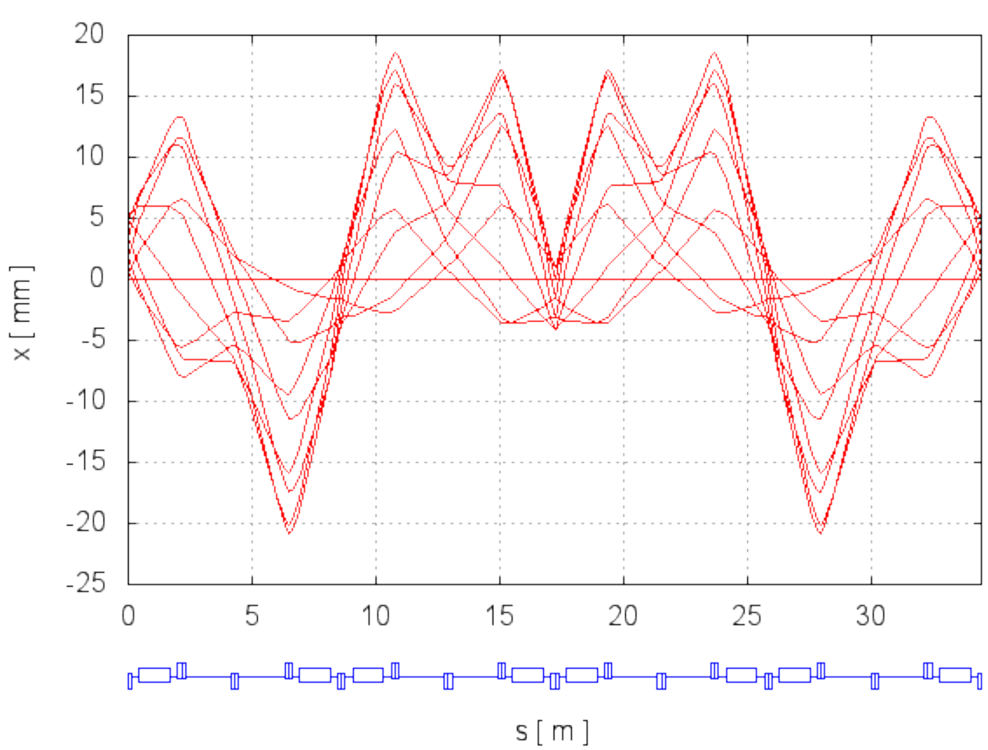
During 2 turns



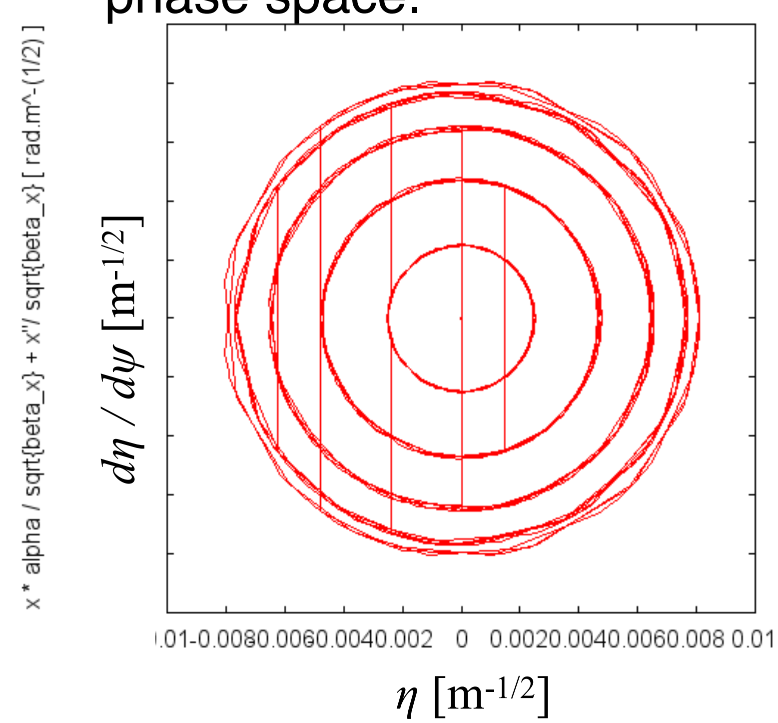
Trajectory at every element of the particle in the normalized phase space.



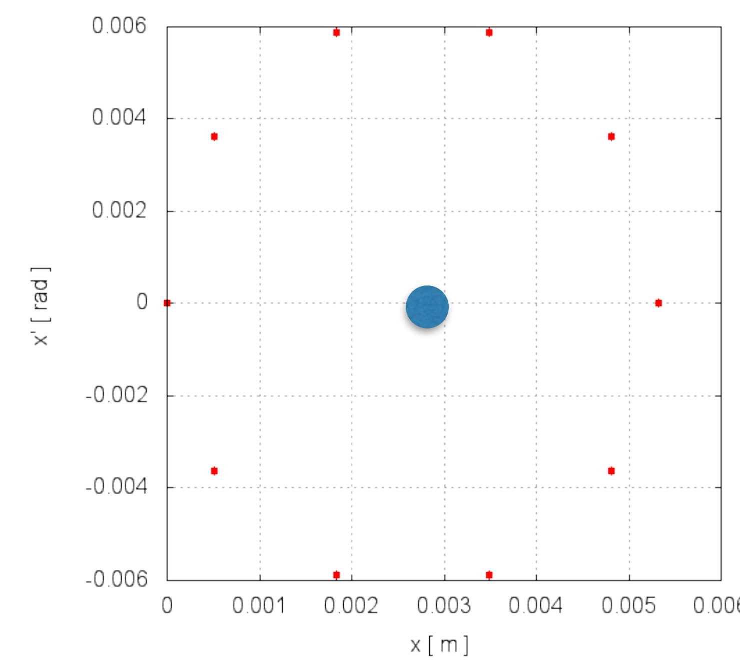
During 10 turns



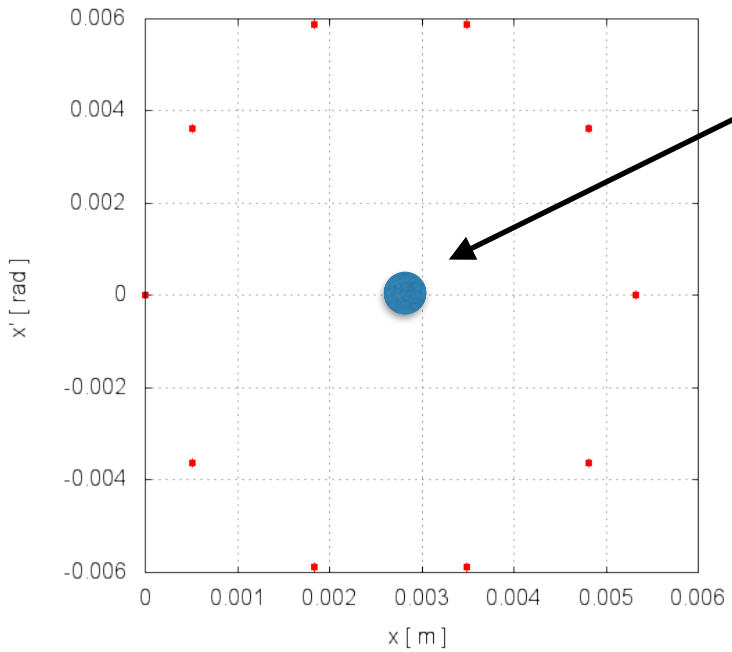
Trajectory at every element of the particle in the normalized phase space.

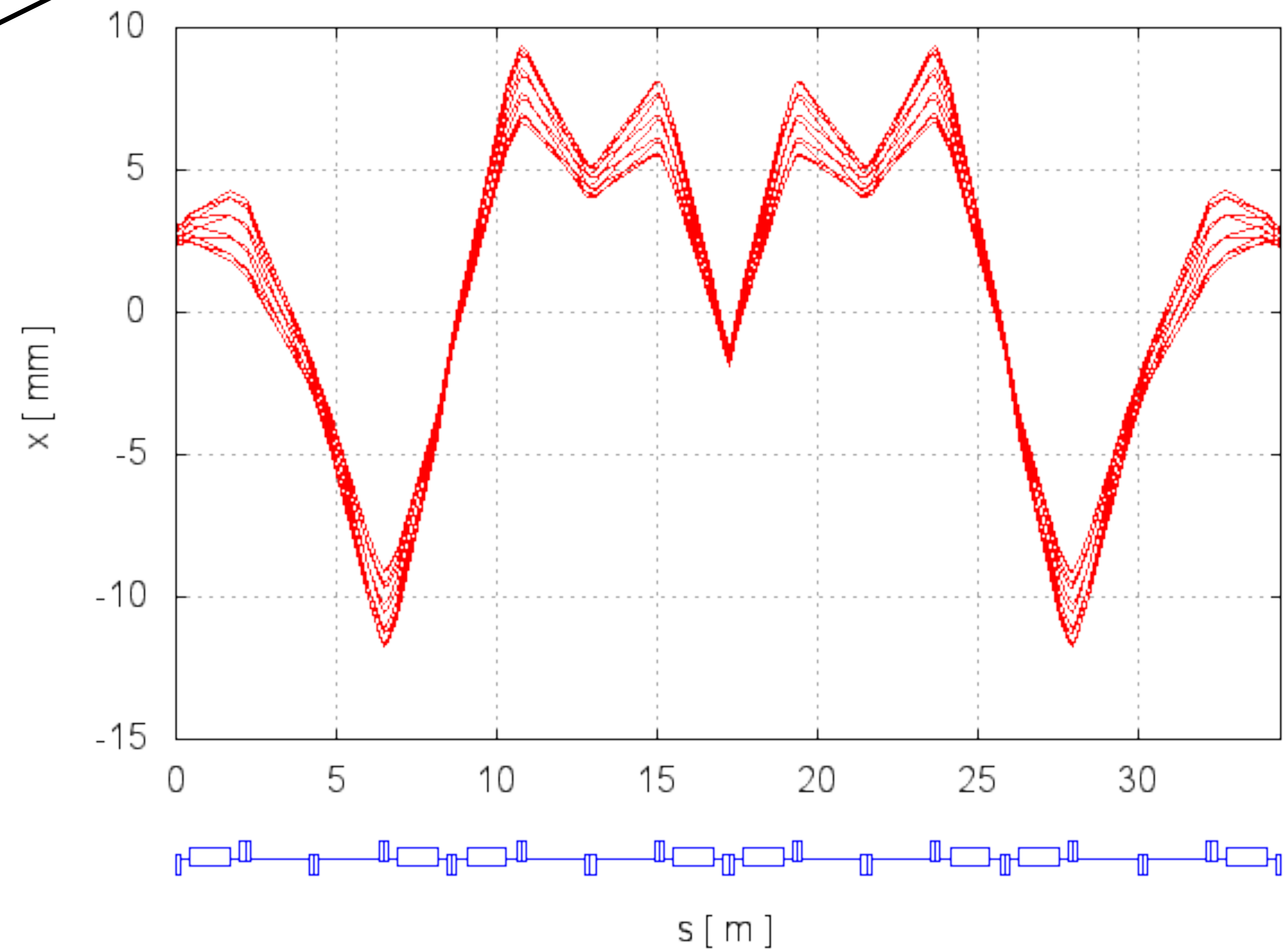


Poincre plot at s=0

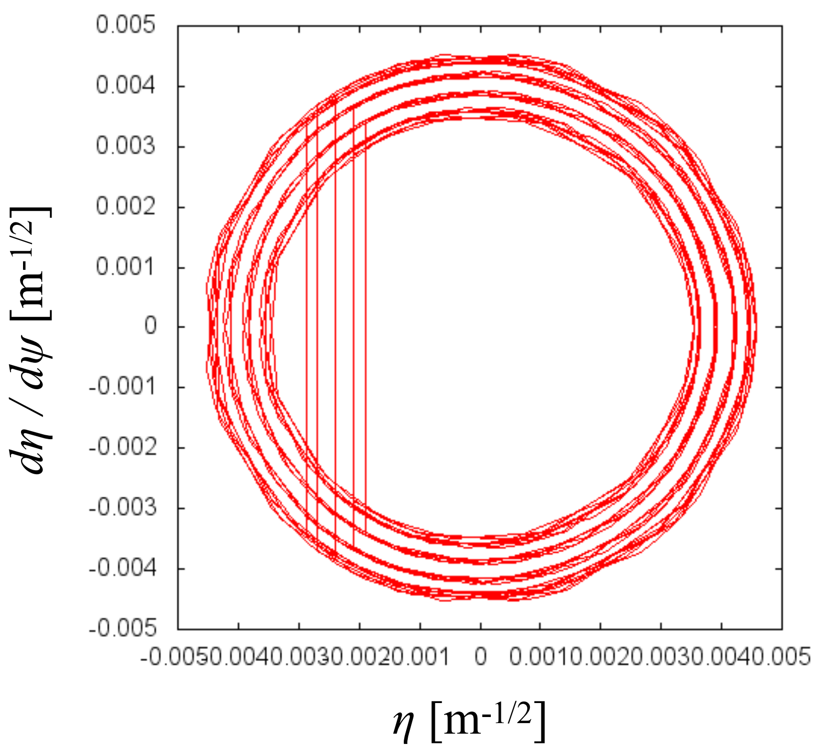


To see the closed orbit behavior, change the launched point.

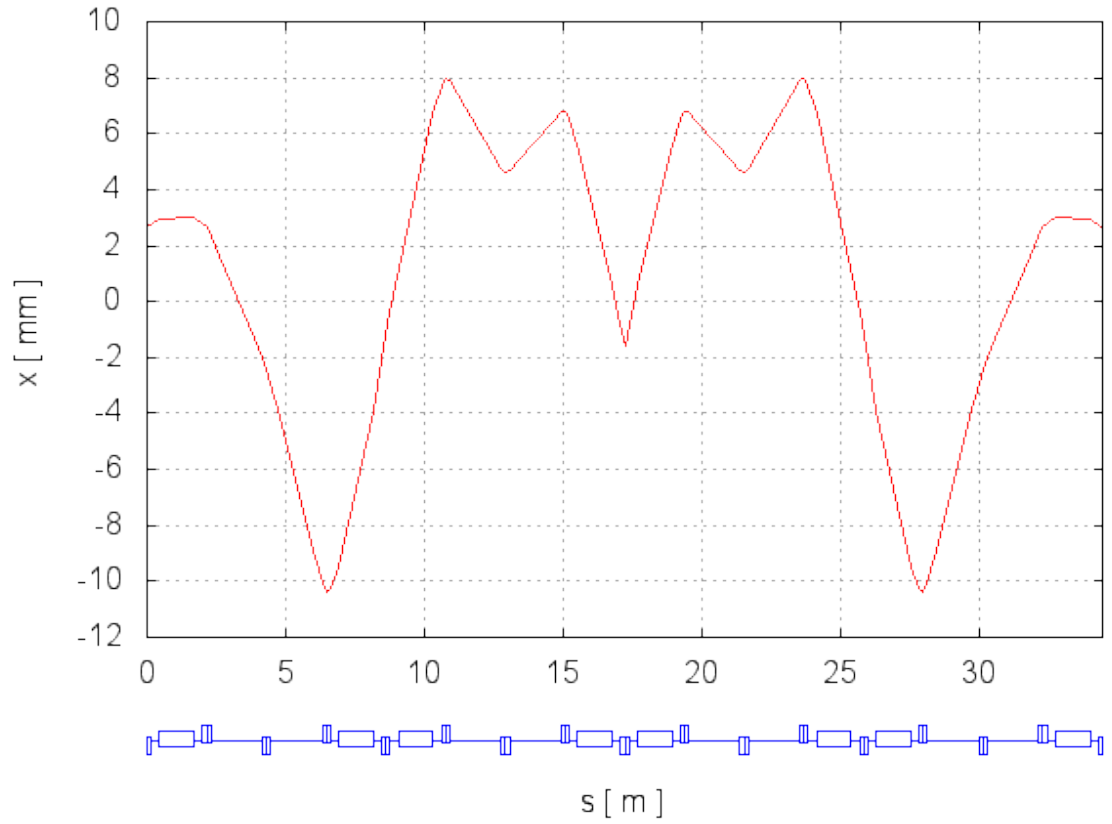


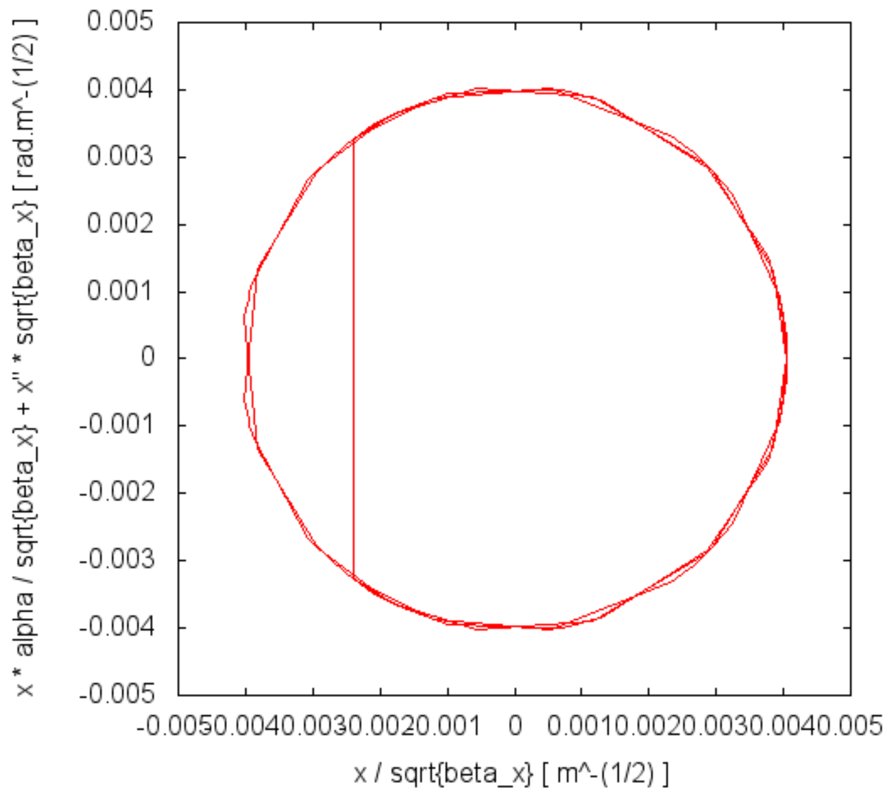


Poincre plot at s=0



In the case with a dipole error, closed orbit is distorted.
This is called COD (closed orbit distortion).





M is one turn transfer matrix at the point where the dipole kick θ is located.

$$M \left(\begin{array}{c} x_0 \\ x_0' \end{array} \right) + \left(\begin{array}{c} 0 \\ \theta \end{array} \right) = \left(\begin{array}{c} x_0 \\ x_0' \end{array} \right)$$

$$\begin{aligned} M &= \left[\begin{array}{cc} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{array} \right] \\ &= I \cos\mu + J \sin\mu \\ &= e^{J\mu} \end{aligned}$$

$$J = \left[\begin{array}{cc} \alpha & \beta \\ -\gamma & -\alpha \end{array} \right]$$

$$J^2 = -I$$

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (I - M)^{-1} \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

$$\begin{aligned}
(I-M)^{-1} &= (I - e^{J\mu})^{-1} = [e^{J\mu/2}(e^{-J\mu/2} - e^{+J\mu/2})]^{-1} \\
&= -(2J\sin\mu/2)^{-1}(e^{J\mu/2})^{-1} \\
&= \frac{1}{2\sin\mu/2} J e^{-J\mu/2} \\
&= \frac{1}{2\sin\mu/2}(J\cos\mu/2 + I\sin\mu/2)
\end{aligned}$$

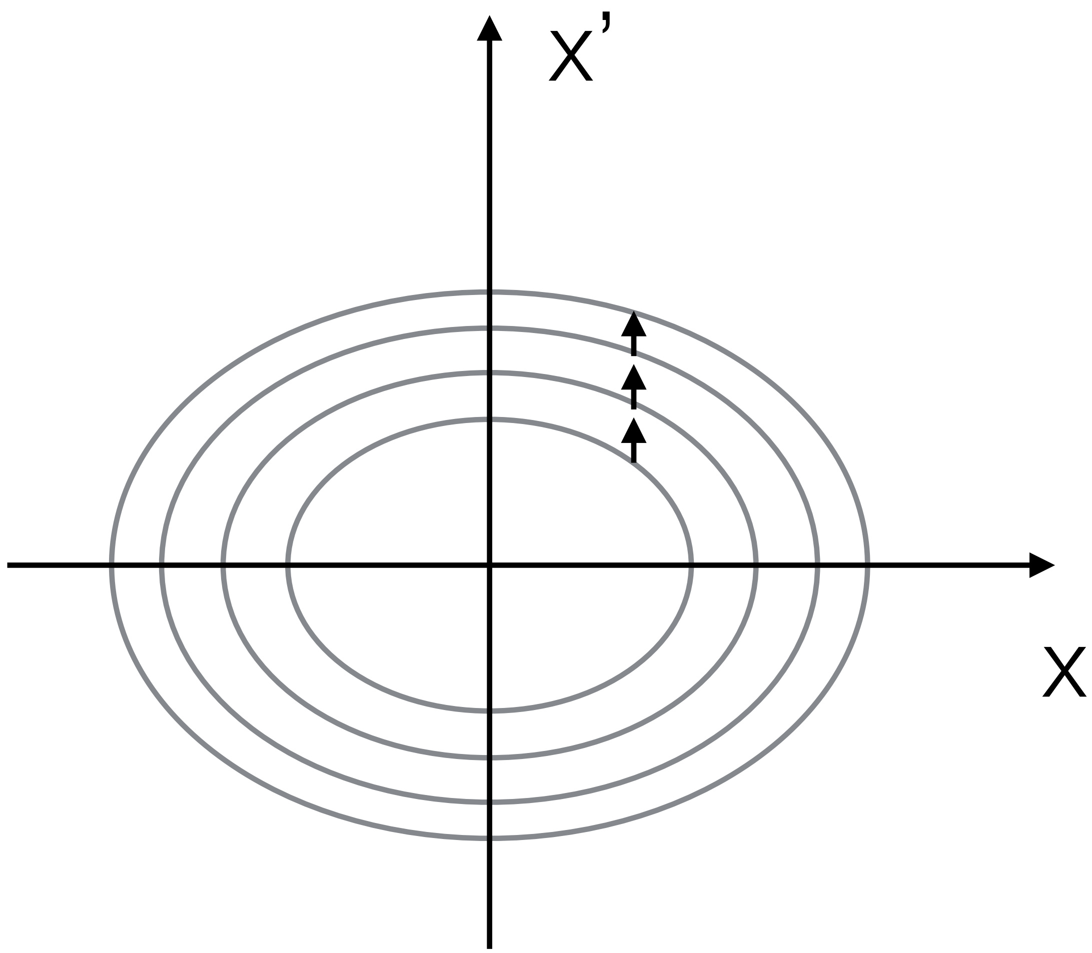
At the point of dipole kick, putting $2\pi\nu = \mu$

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (I - M)^{-1} \begin{pmatrix} 0 \\ \theta \end{pmatrix} = \frac{\theta}{2\sin\pi\nu} \begin{pmatrix} \beta_0 \cos\pi\nu \\ \sin\pi\nu - \alpha_0\cos\pi\nu \end{pmatrix}$$

The displacement x(s) at an arbitrary point s.

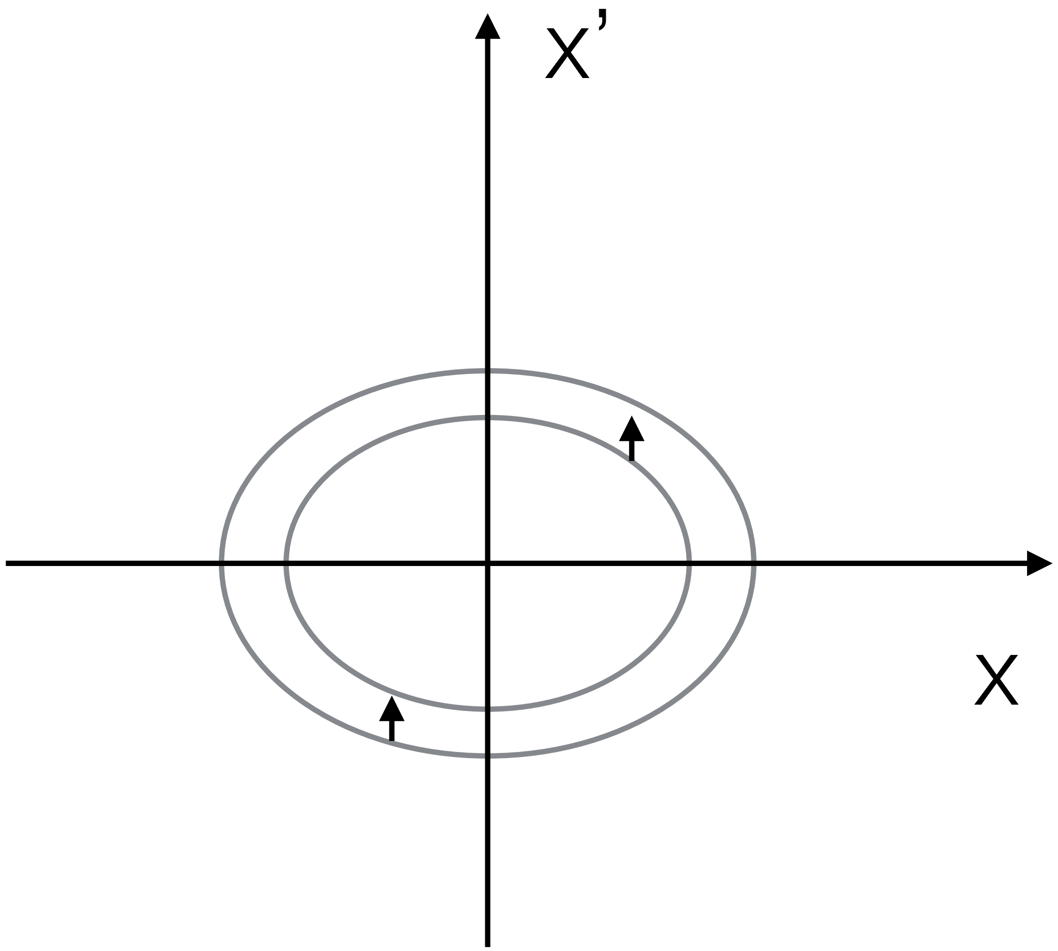
$$x(s) = \frac{\theta\sqrt{\beta(s)\beta_0}}{2\sin\pi\nu}\cos[\psi(s) - \pi\nu]$$

If the tune is closed to the integer, the denominator tends to zero and COD becomes infinity.

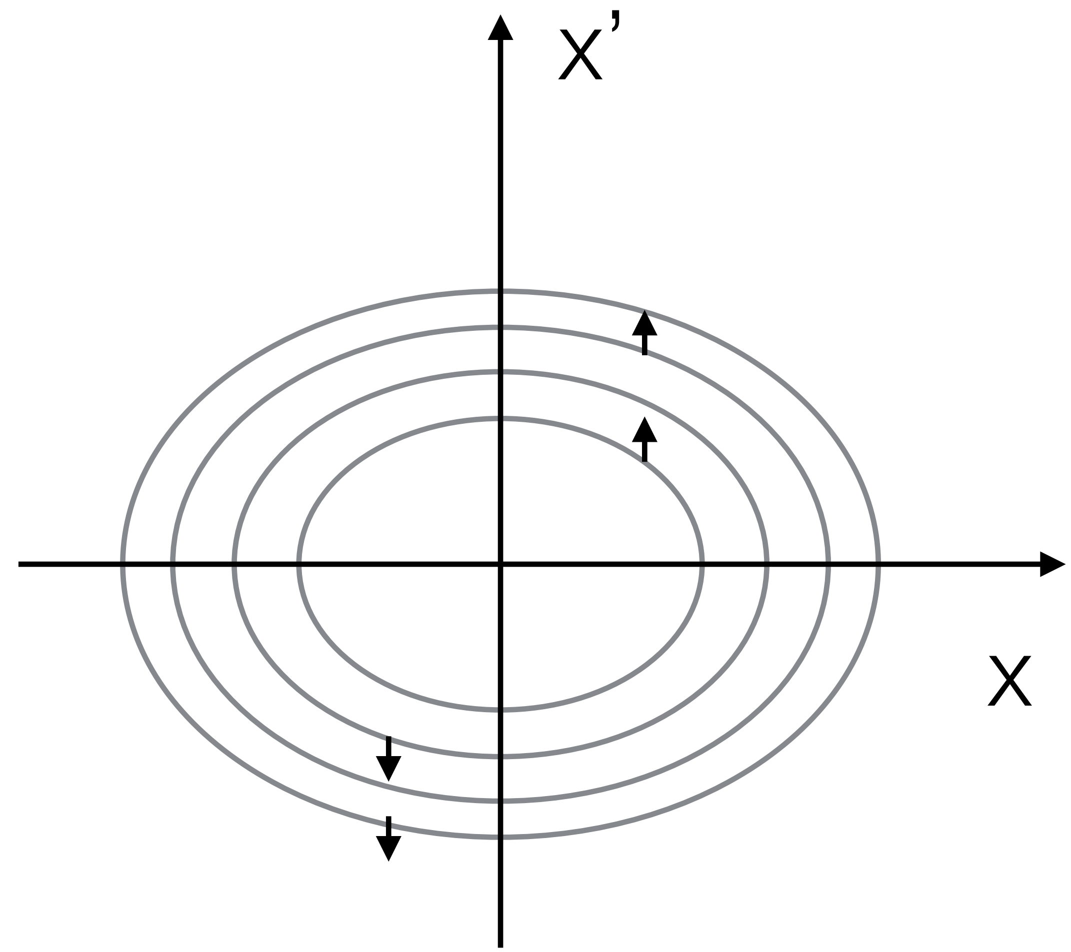


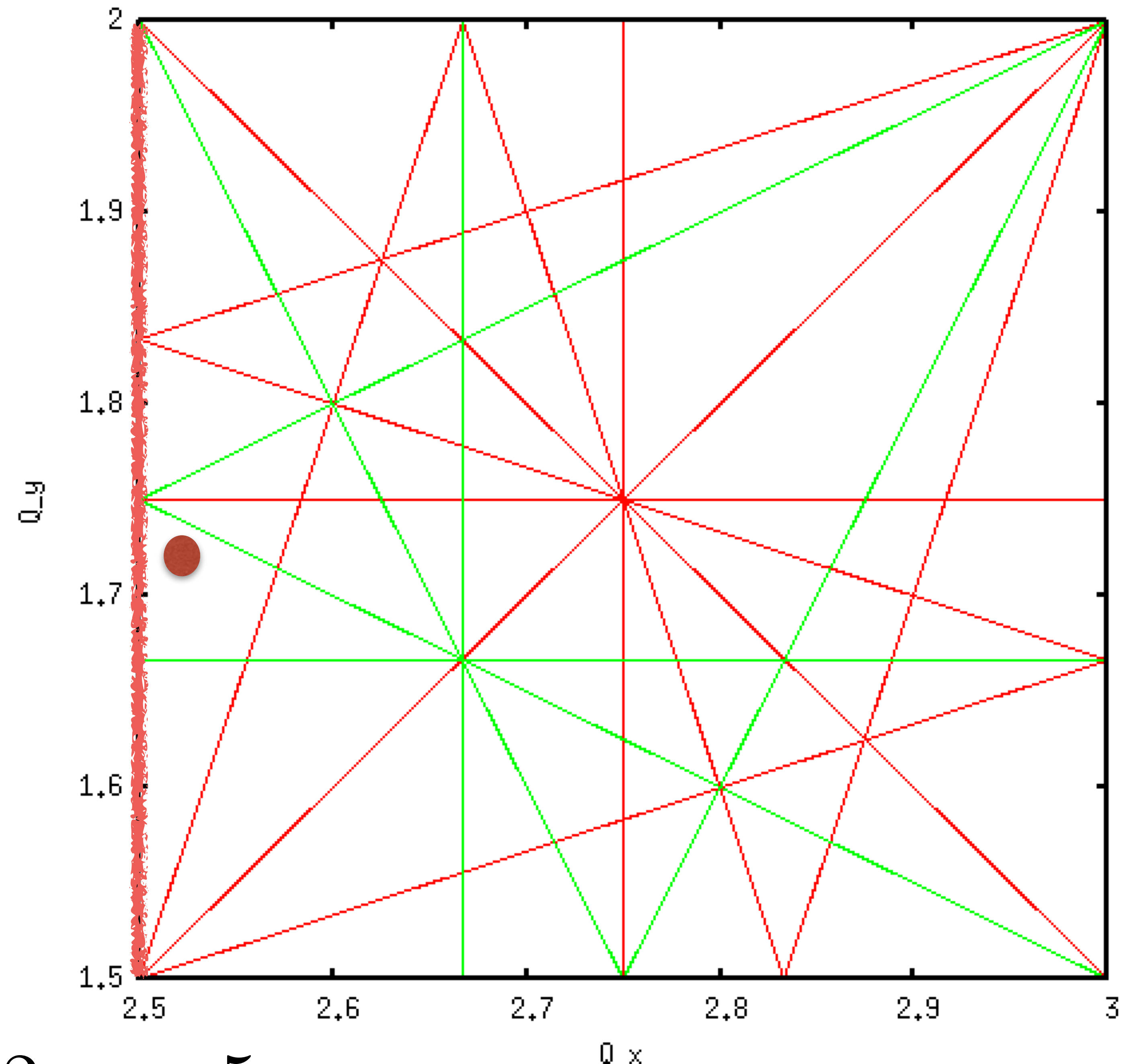
Quadrupole error and the half integer resonance

If the tune is closed to half integer, the error dipole kicks from two consecutive orbital revolutions cancel each other.



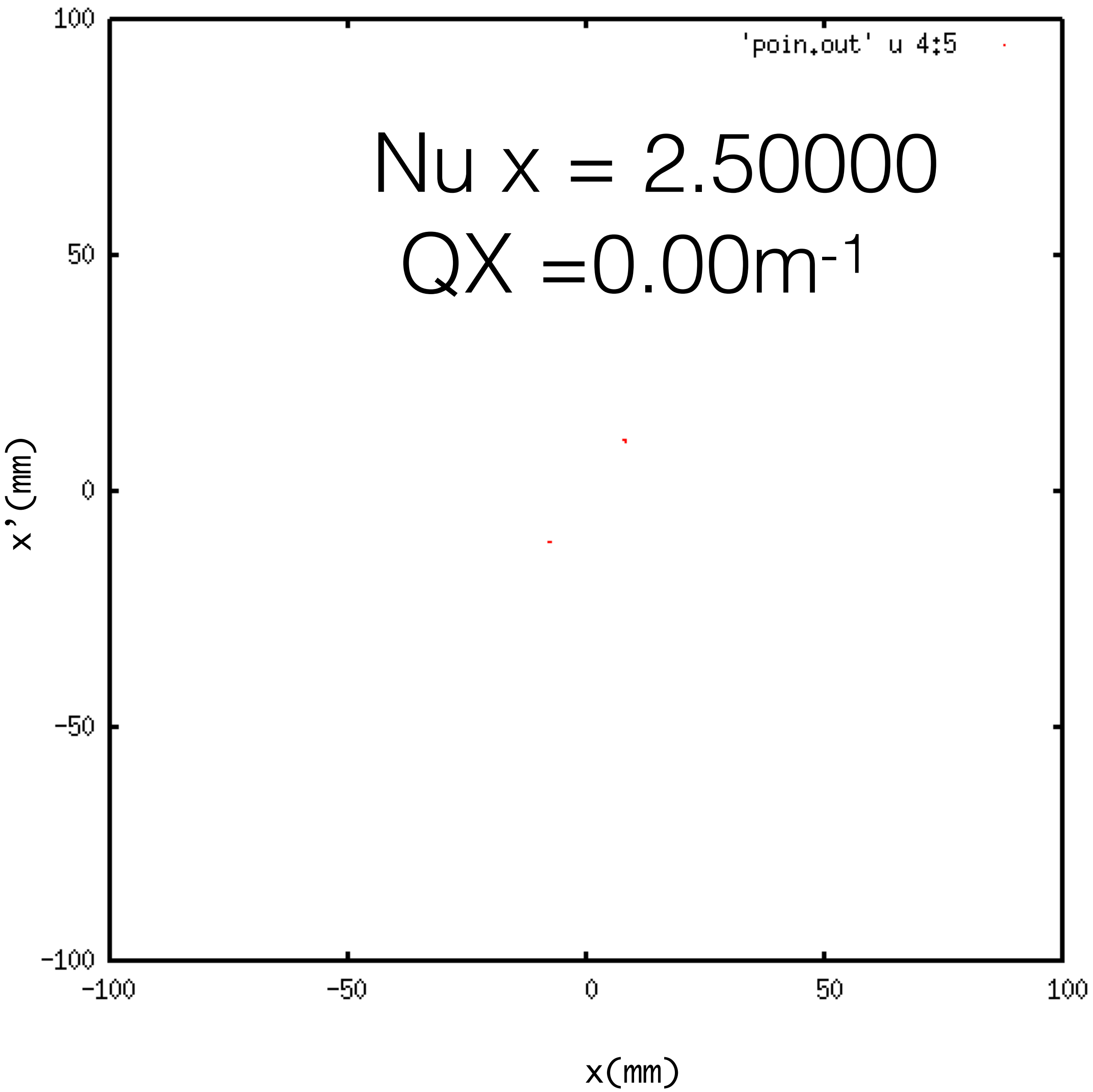
If the tune is closed to half integer, quadrupole error blows up the amplitude.



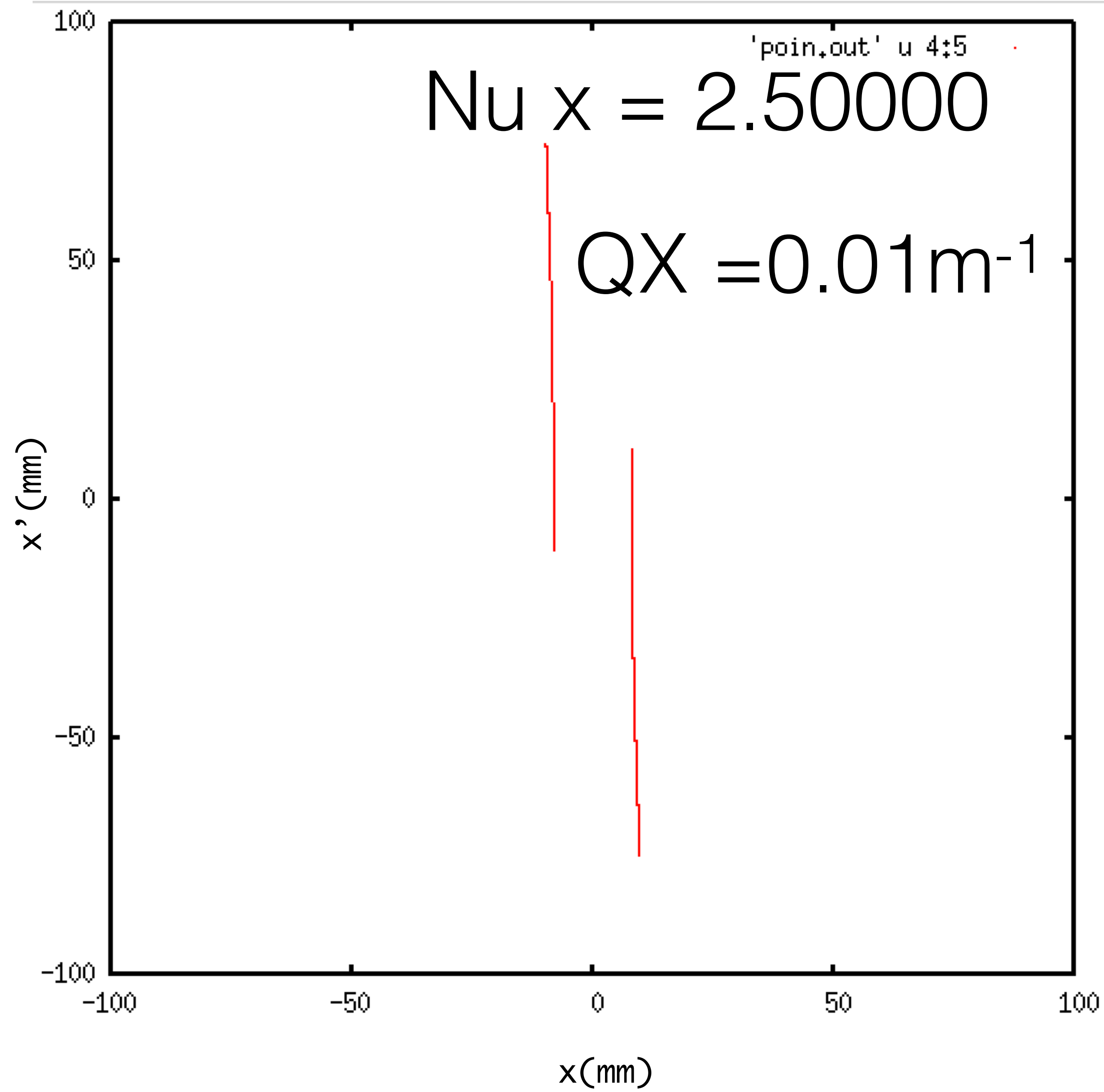


$$2\nu_x = 5$$

Nux = 2.50000



'poin.out' u 4:5
Nu x = 2.50000
QX =0.00m-1
x'(mm)
x(mm)
100
50
0
-50
-100
-100
-50
0
50
100



'poin.out' u 4:5
Nu x = 2.50000
QX =0.01m-1
x'(mm)
x(mm)
100
50
0
-50
-100
-100
-50
0
50
100

Similar to the case of dipole error, M is one turn transfer matrix at the point where the quadrupole error is located. Error is regarded thin lens whose focal length is $1/k$. Error free transfer matrix M_0 and one with error M should be expressed as

$$
\begin{aligned}
M &= M_0 \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\mu_0 + \alpha\sin\mu_0 & \beta\sin\mu_0 \\ -\gamma\sin\mu_0 & \cos\mu_0 - \alpha\sin\mu_0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\mu_0 + \alpha\sin\mu_0 - k\beta\sin\mu_0 & \beta\sin\mu_0 \\ -\gamma\sin\mu_0 - k(\cos\mu_0 - \alpha\sin\mu_0) & \cos\mu_0 - \alpha\sin\mu_0 \end{bmatrix}
\end{aligned}
$$

$2\pi\nu_0 = \mu_0$ and from the trace of M $\cos 2\pi\nu$ should be

$$\cos 2\pi\nu = \cos 2\pi\nu_0 - \frac{1}{2}k\beta\sin 2\pi\nu_0$$

expanding the left hand side with $\nu = \nu_0 + \Delta\nu$

$$\cos 2\pi\nu = \cos 2\pi\nu_0 - 2\pi\Delta\nu \sin 2\pi\nu_0$$

Therefor, the tune is shifted by quadrupole error by

$$\Delta\nu = \frac{1}{4\pi} k\beta$$

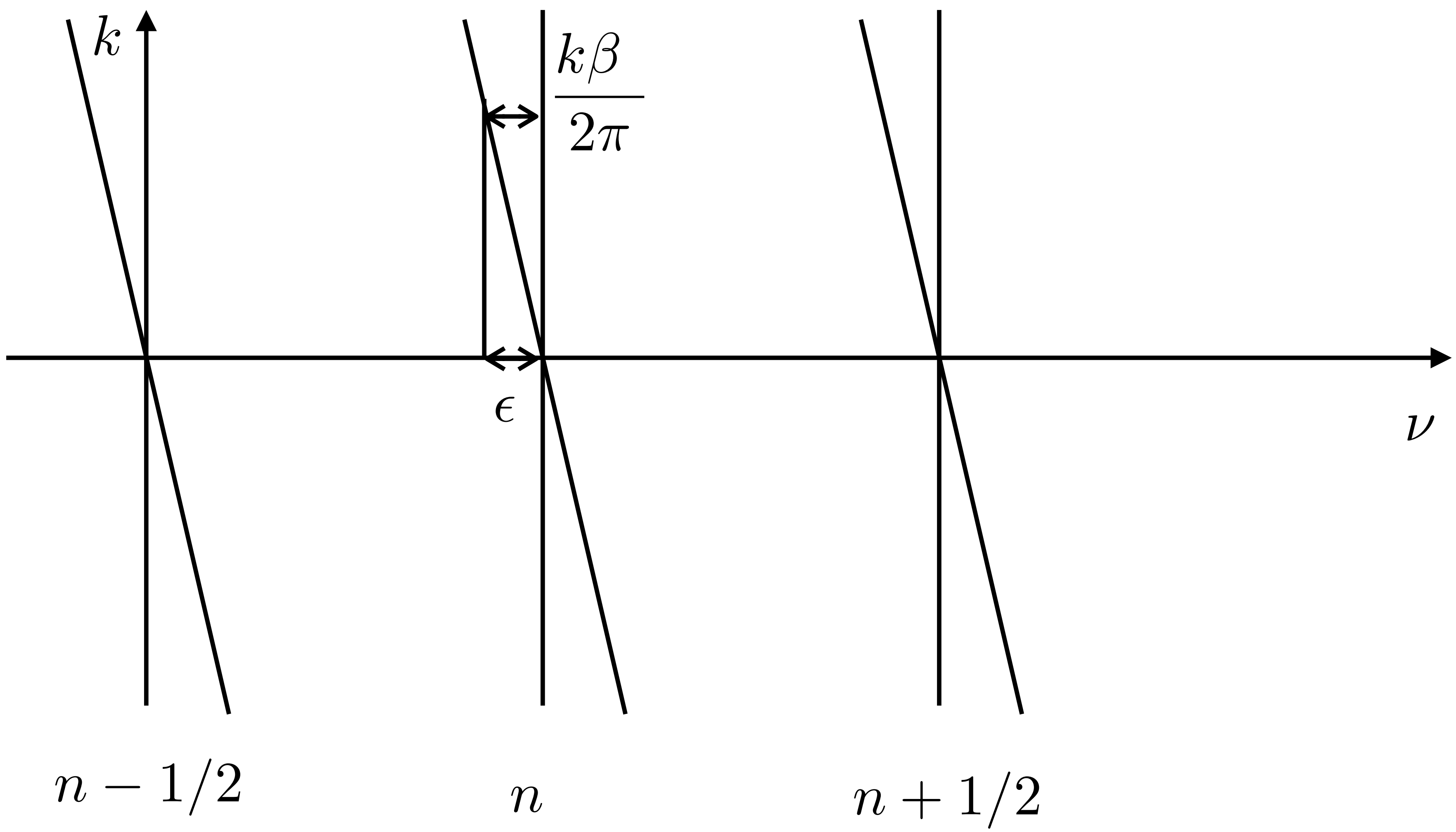
For the case that the tune is closed to an integer $\tilde{\nu} + \epsilon$

$$\begin{aligned} \cos 2\pi\nu &= \cos 2\pi\tilde{\nu} - 2\pi\epsilon \sin 2\pi\tilde{\nu} - \frac{(2\pi\epsilon)^2}{2}\cos^2 2\pi\tilde{\nu} \\ &\quad -\frac{1}{2}k\beta(\sin 2\pi\tilde{\nu} + 2\pi\epsilon\cos 2\pi\tilde{\nu}) \\ &= 1 - 2\pi^2\epsilon^2 - k\beta\pi\epsilon \end{aligned}$$

for the stable solution (tune stays real), the condition is

$$\epsilon \left(\epsilon + \frac{k\beta}{2\pi} \right) > 0 \quad .$$

stop band

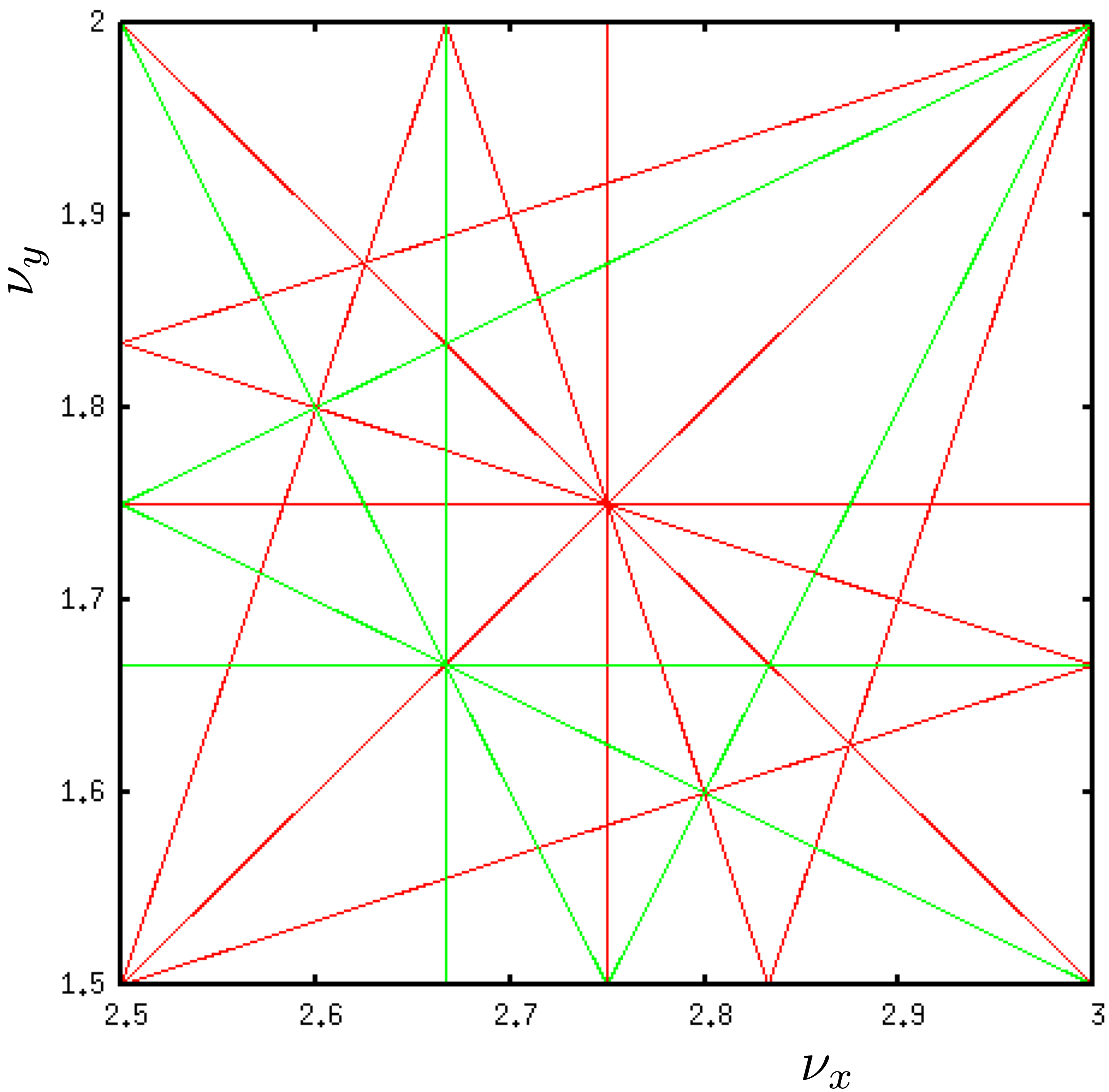


Resonance condition in an accelerator

$$k\nu_x + m\nu_y = n$$

k, m, n are positive or negative integer

$$|k| + |m| = \text{order}$$



order =1 : Integer resonance

by dipole

$\nu_x = n$ $\nu_y = n$

order = 2 : Half-integer resonance

by normal quadrupole

$2\nu_x = n$ $2\nu_y = n$

by skew quadrupole

$\nu_x + \nu_y = n$

sum resonance

$\epsilon_x - \epsilon_y = C$

$\nu_x - \nu_y = n$

difference resonance

$\epsilon_x + \epsilon_y = C$

C: constant

order = 3: Third-integer resonance

by normal sextupole

$3\nu_x = n, \nu_x \pm 2\nu_y = n$

by skew sextupole

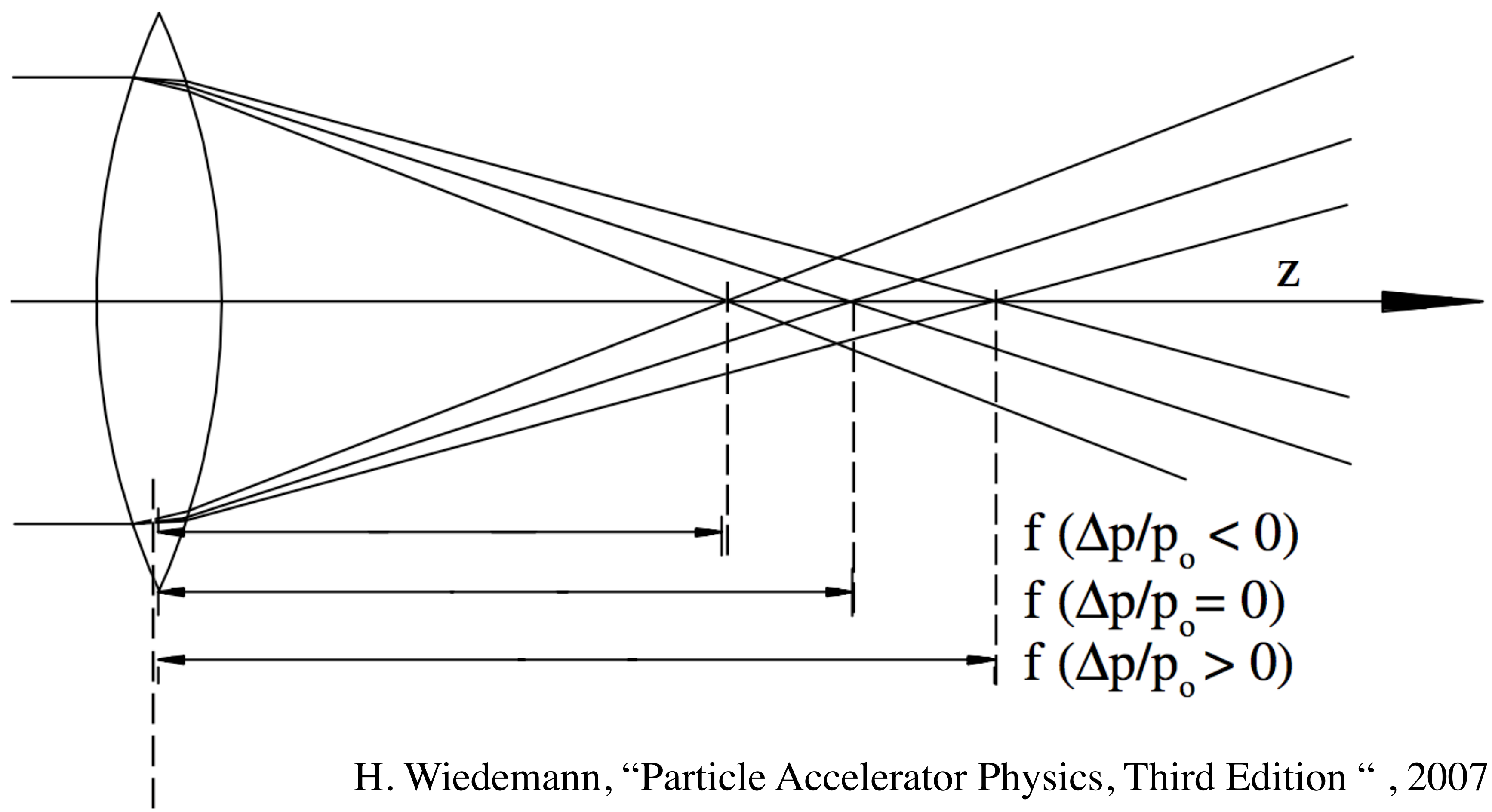
$3\nu_y = n, 2\nu_x \pm \nu_y = n$

How come non-linear is brought into the ring?

1. To correct the chromaticity
2. Magnet imperfection
3. To extract the beam blowing up the amplitude due to the third-integer resonance.

Chromaticity

chromaticity is an aberration in the accelerator i.e. tune shift.



H. Wiedemann, “Particle Accelerator Physics, Third Edition “ , 2007, Springerより

Quadrupole magnet with field gradient of g gives the focusing force of K to the particle with momentum of p and charge e.

$$K = \frac{\partial B_y}{\partial x}\frac{1}{B\rho} = \frac{eg}{p}$$

for the particle with higher momentum by Δp, replacing $p \to p + \Delta p$

$$K \to K\frac{p}{p+\Delta p} = K\frac{1}{1+\frac{\Delta p}{p}} \sim K - \boxed{K\frac{\Delta p}{p}}$$

The focusing force is reduced by this amount. This can be regarded as quadrupole error.

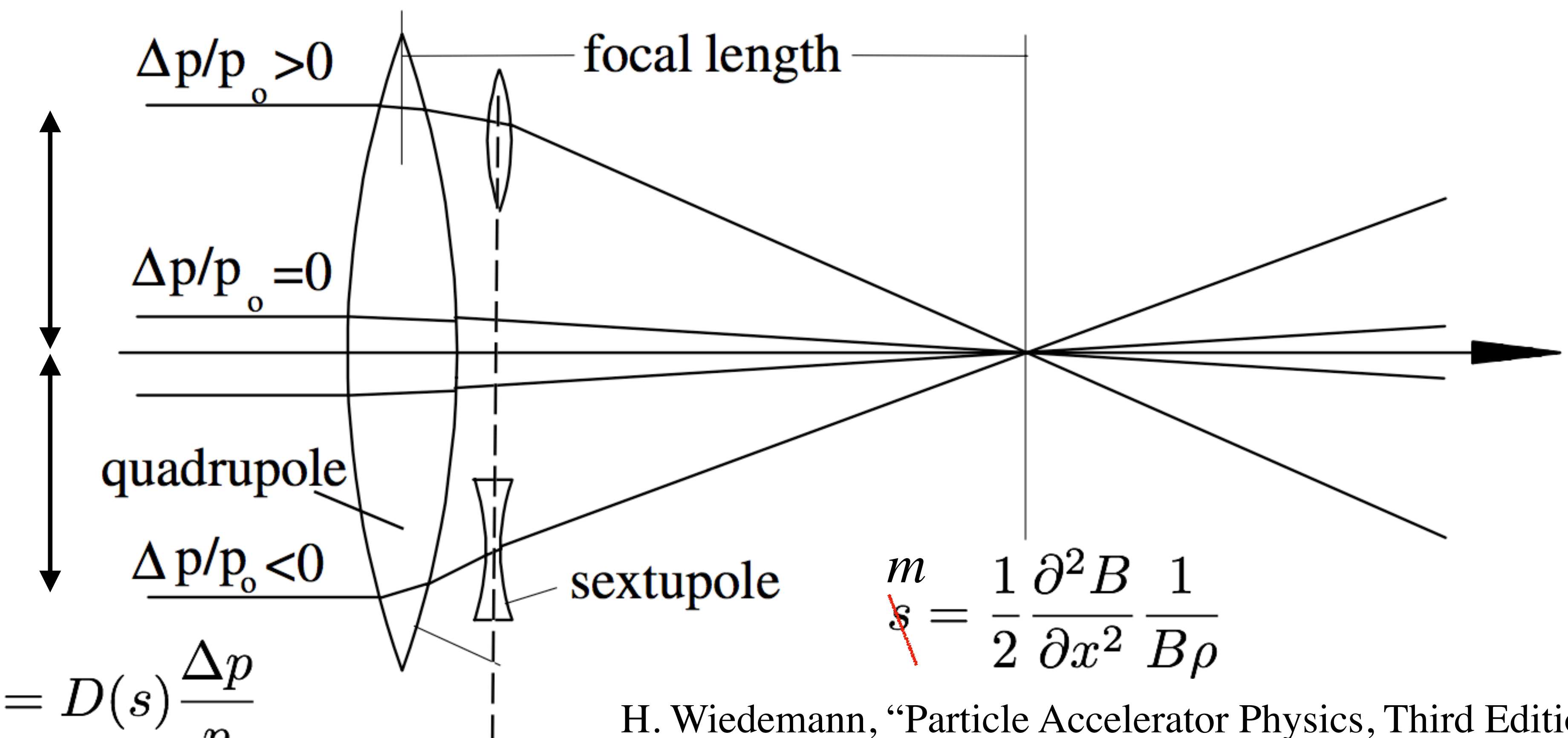
$$\Delta\nu = -\frac{1}{4\pi}\oint \beta(s)\Delta K(s)ds = -\frac{1}{4\pi}\oint \beta(s)K(s)ds\frac{\Delta p}{p}$$

$$\Delta\nu = \xi\frac{\Delta p}{p}$$

ξ is called chromaticity and defined as

$$\xi = -\frac{1}{4\pi}\oint \beta(s)K(s)ds$$

tune shift → tune satisfies the resonance conditions→beam loss occurs → needs correction using sextuple magnets



$$m = \frac{1}{2}\frac{\partial^2 B}{\partial x^2}\frac{1}{B\rho}$$

$$\Delta x(s) = D(s)\frac{\Delta p}{p}$$

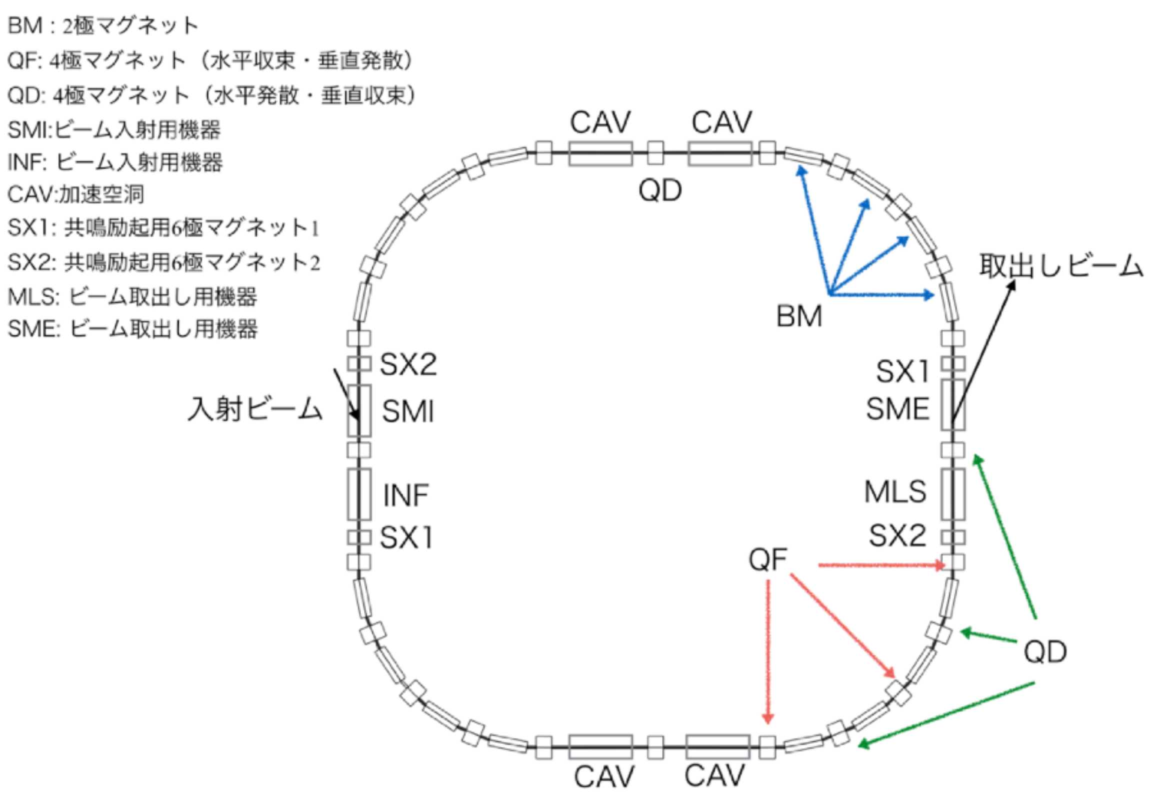
H. Wiedemann, “Particle Accelerator Physics, Third Edition “ , 2007, Springerより

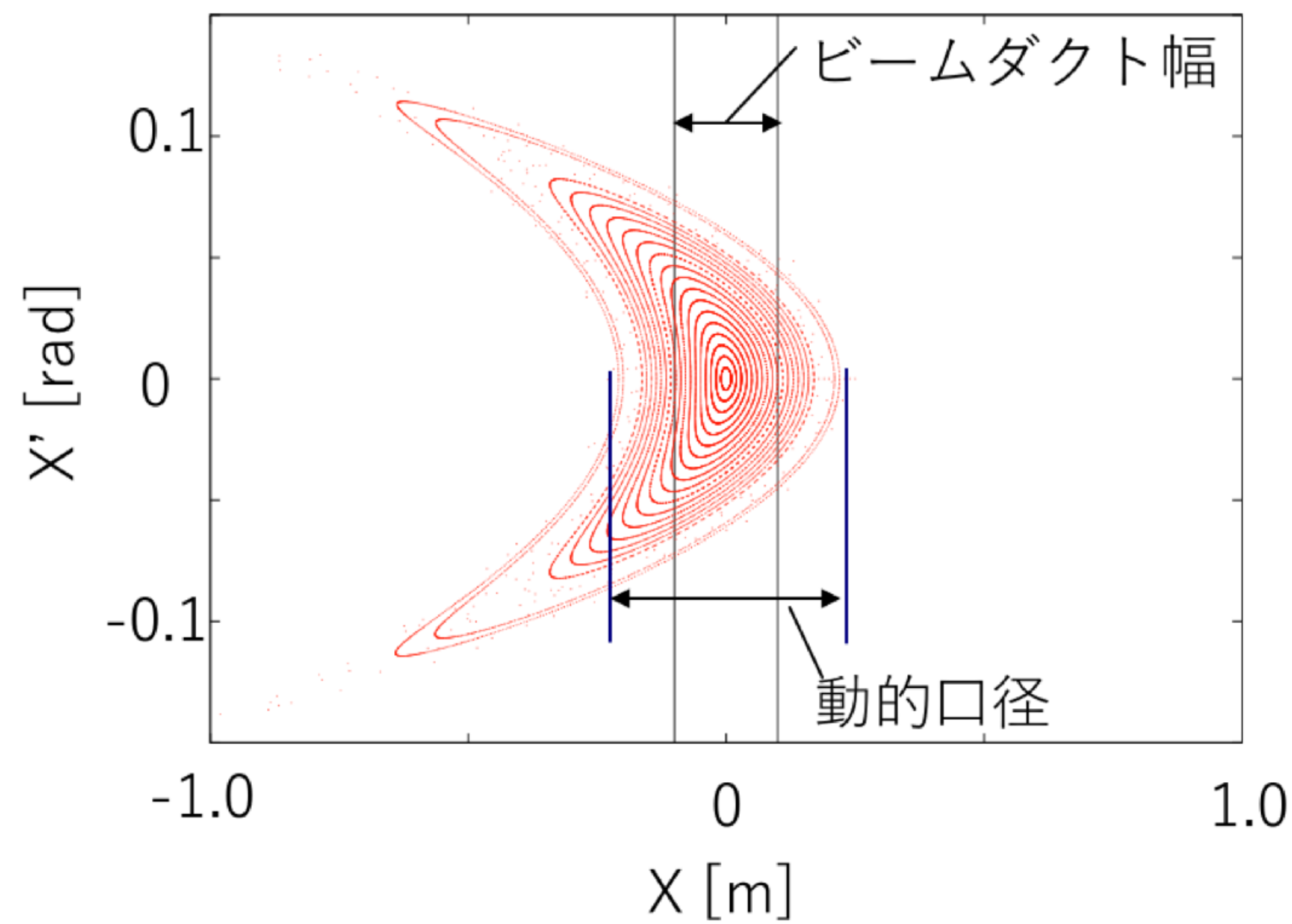
補正後のchromaticityは下記の様になる。

$$\xi_x = \frac{1}{4\pi}\oint (2mD(s) - K)\beta_x(s)ds$$

$$\xi_y = \frac{1}{4\pi}\oint (K - 2mD(s))\beta_y(s)ds$$

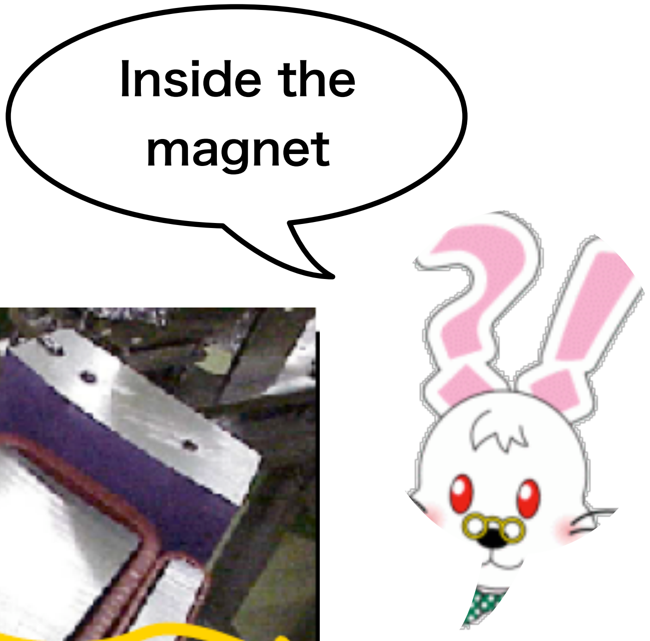
After chromaticity correction, non-linearity reduces dynamic aperture

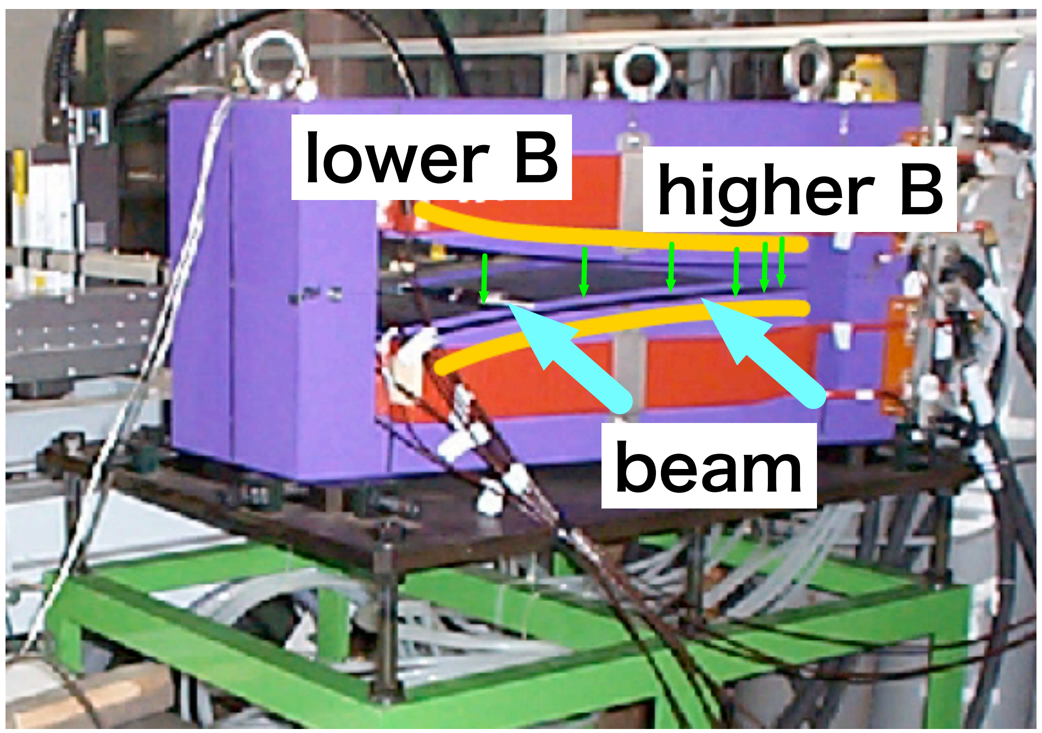




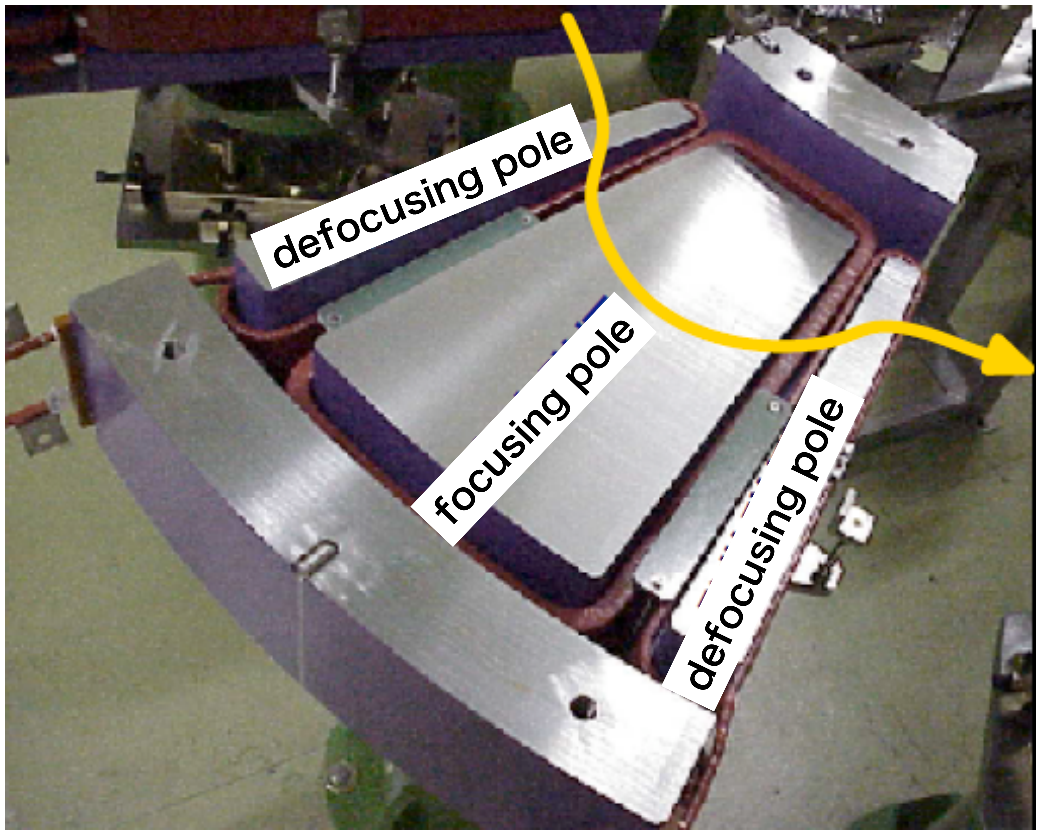
Go back to FFAG

If you want to use fixed field in strong focusing, you need reverse bending.





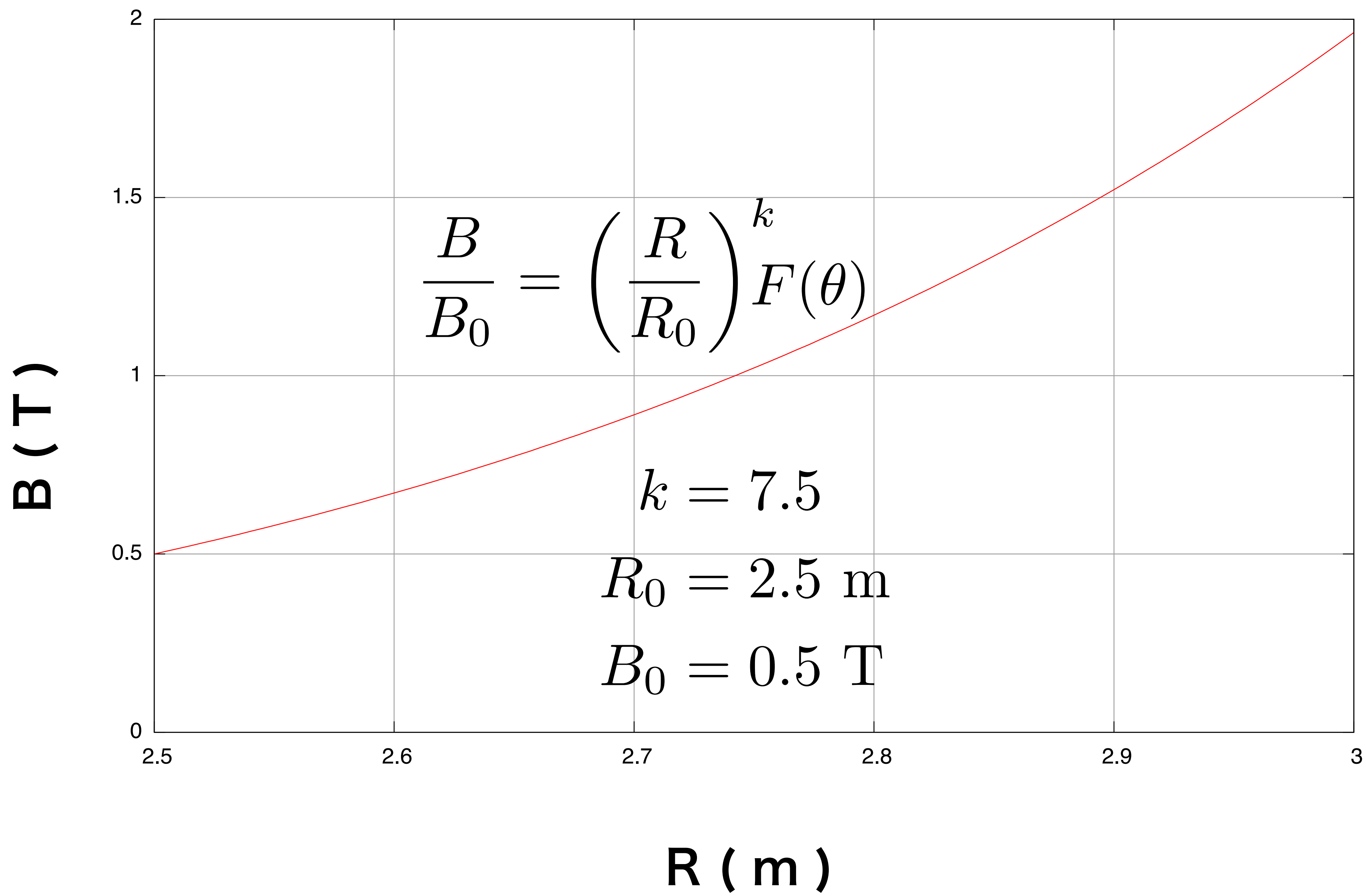
Outer radii the stronger B field



Reverse bend generates focusing force in vertical direction.

Advantage of FFAG

Zero chromaticity

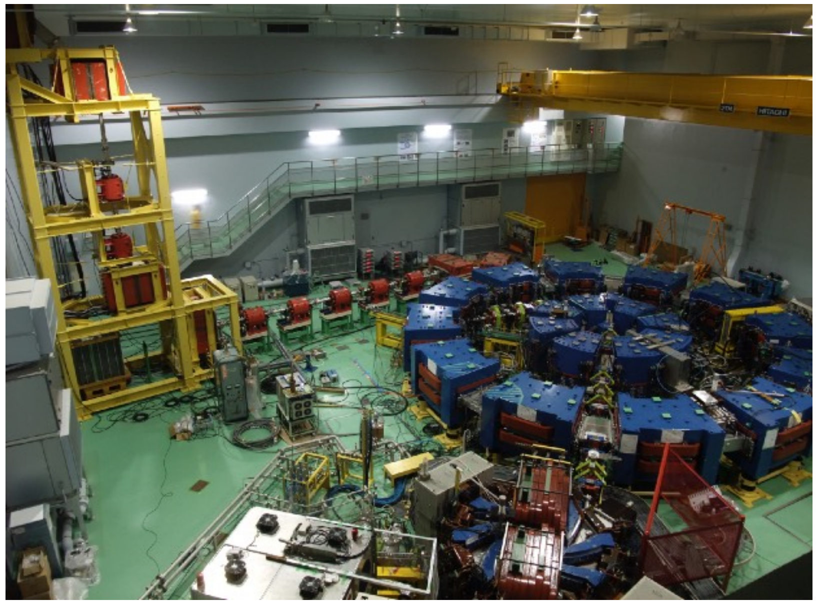


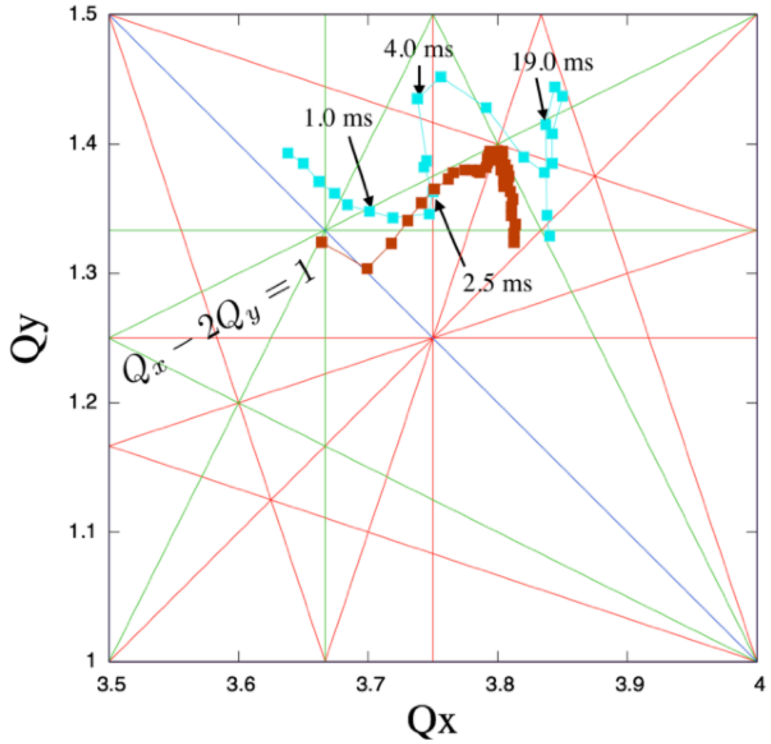
$$k = \frac{R}{B}\left(\frac{\partial B}{\partial R}\right)$$

Advantage of FFAG

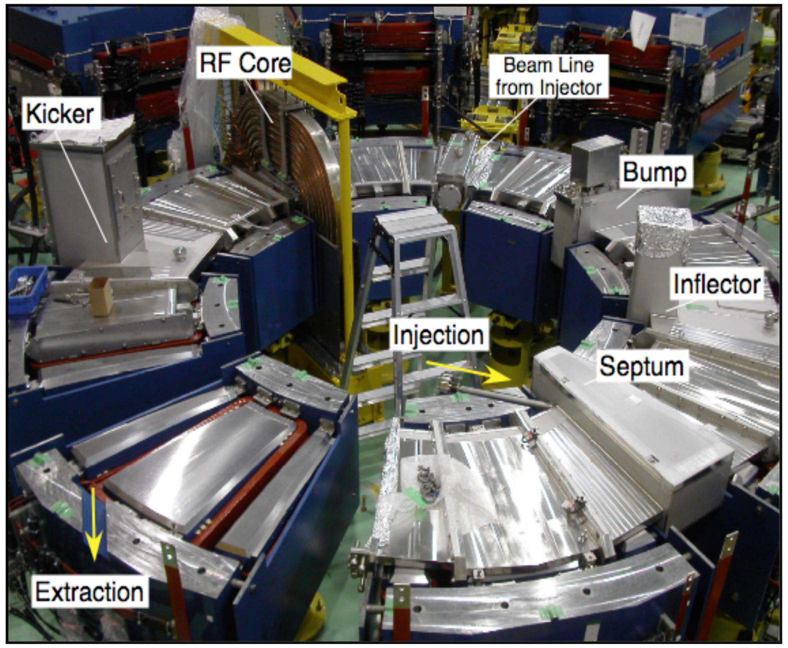
Avoid resonance crossing

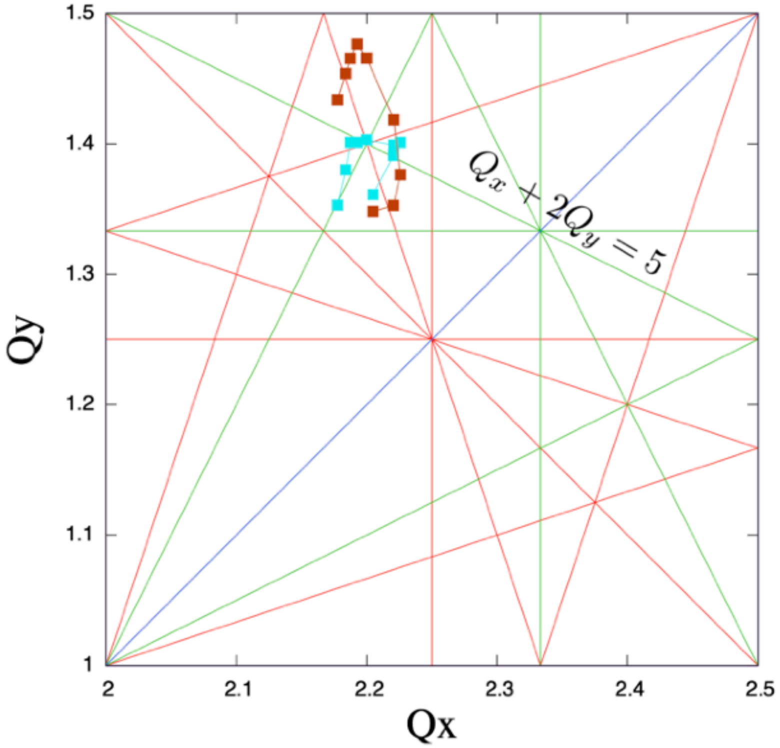
KURNS main ring





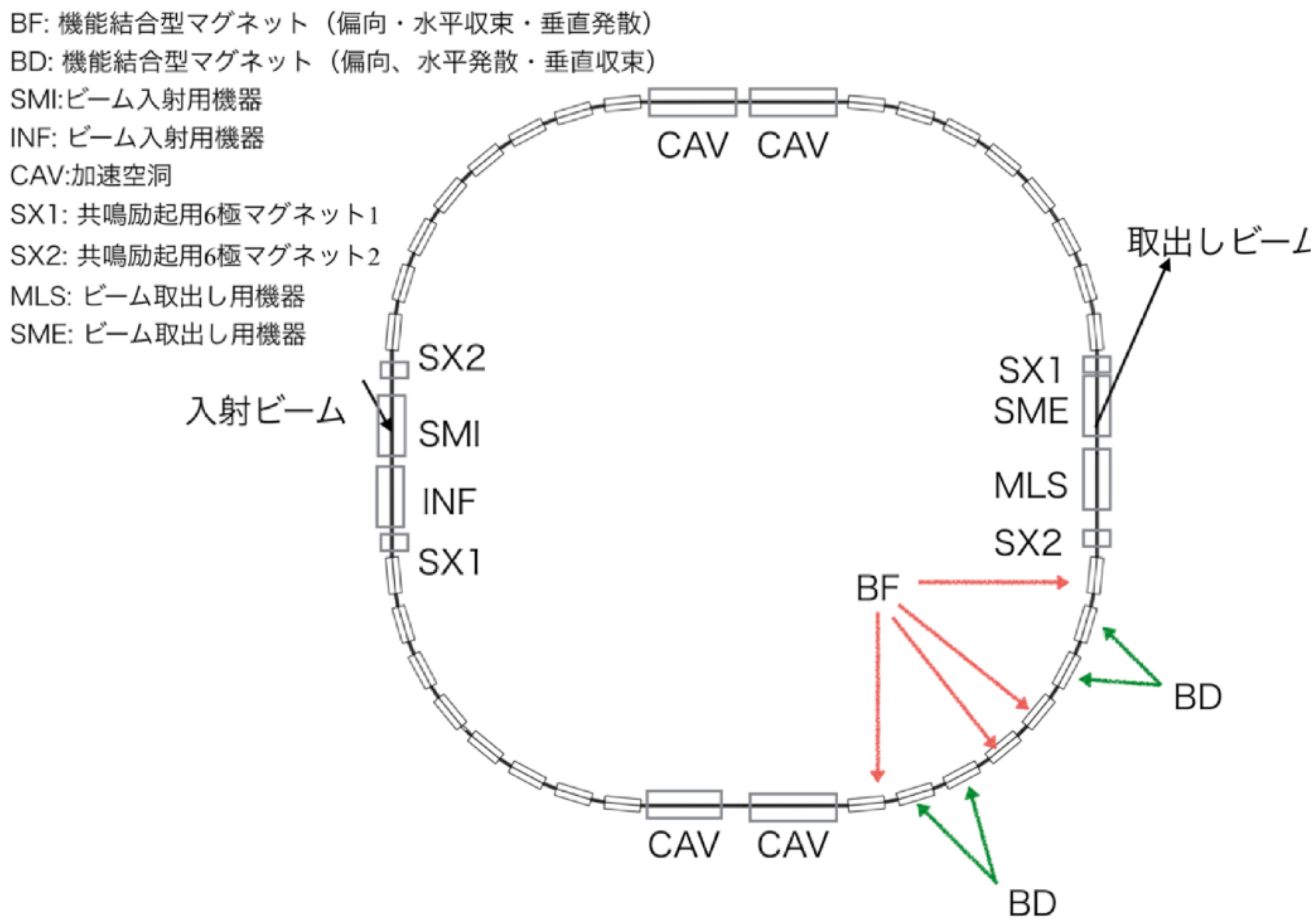
KURNS booster

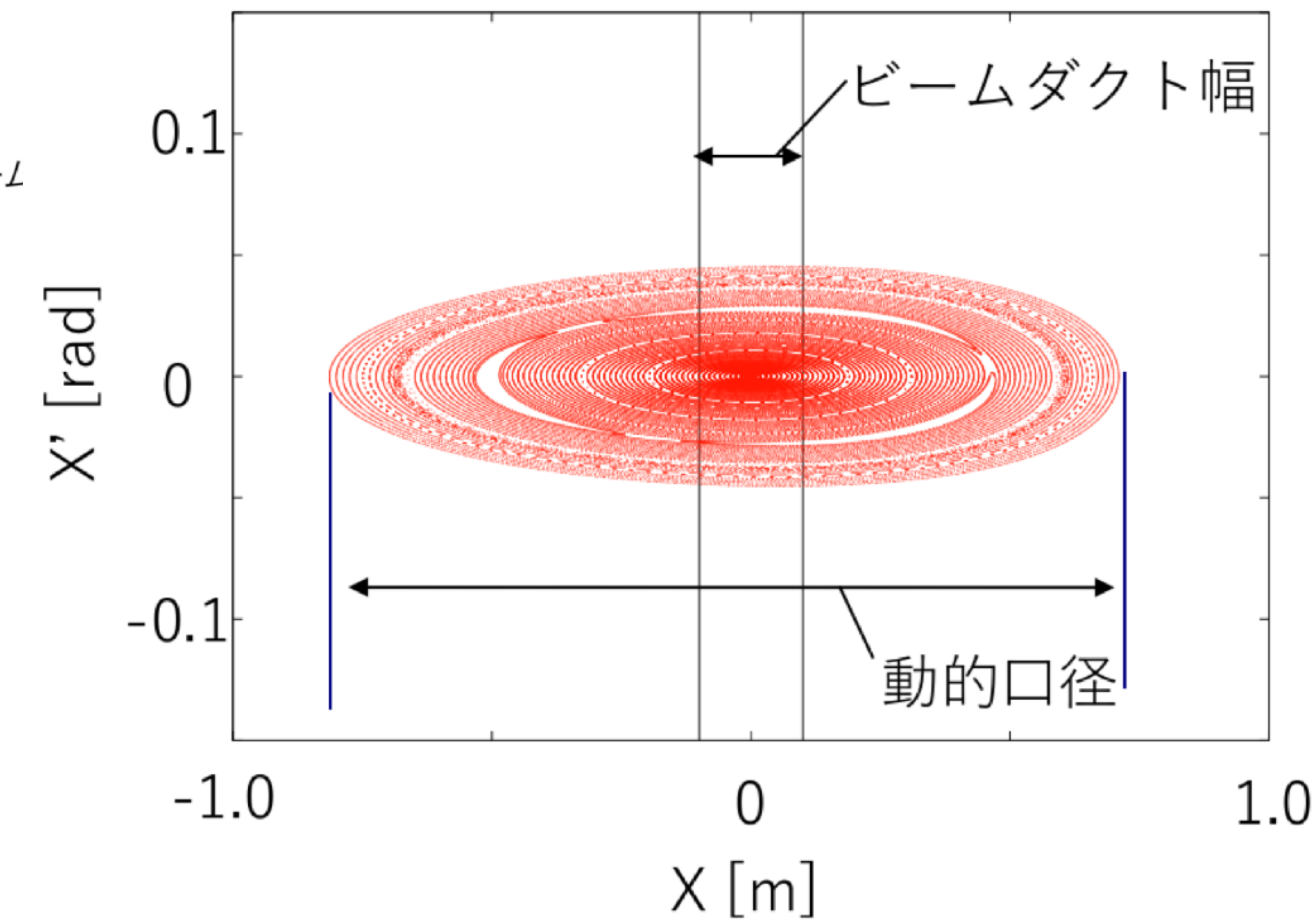




Advantage of FFAG

Large dynamic aperture





Non-linearity is included in main magnet and speed out wide and thin.

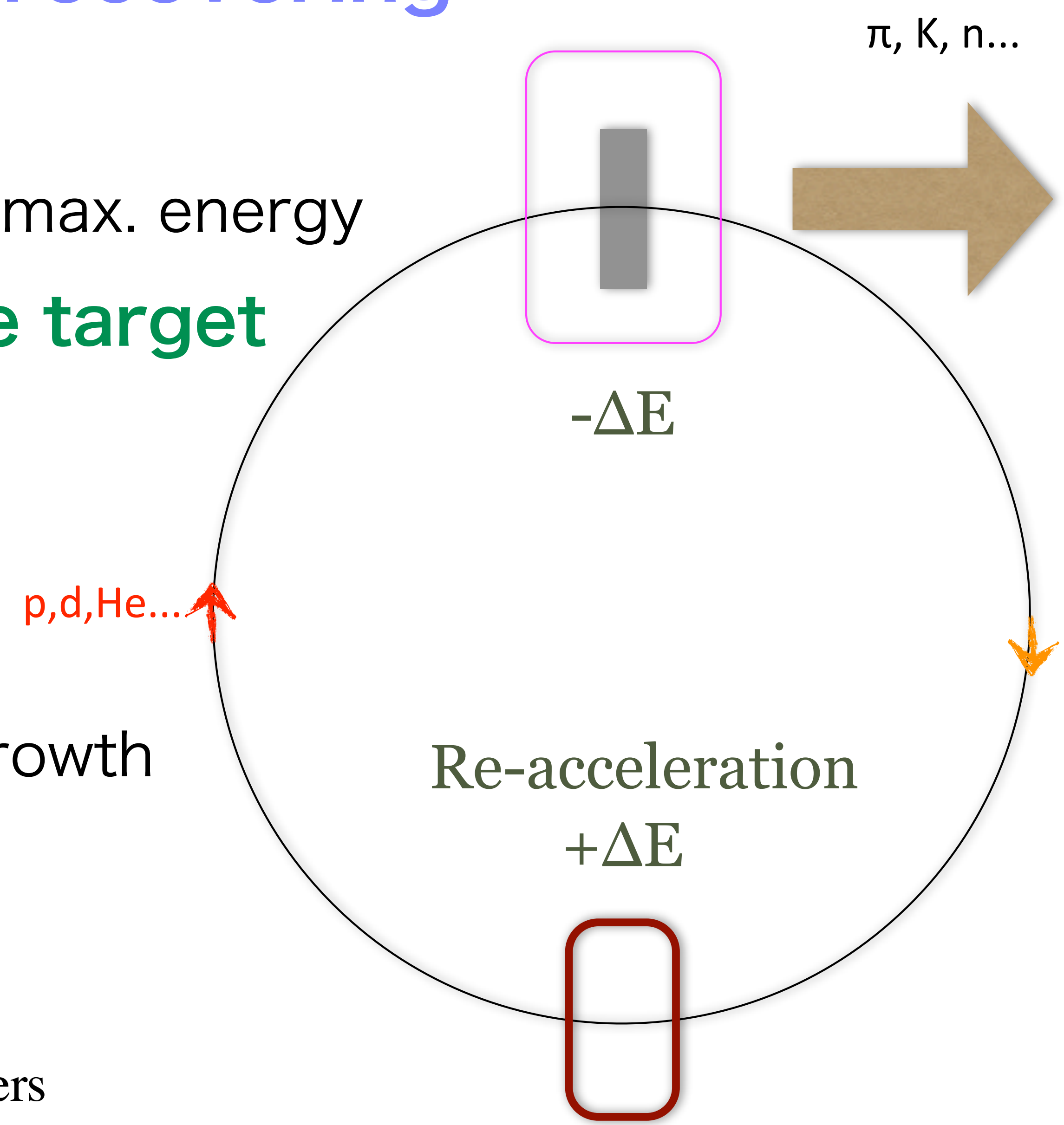
Advantage of FFAG

secondary particle production in storage mode

ERIT:Energy Recovery Internal Target

Thin target & energy recovering

- **No energy degradation**
  - reactions occur always at max. energy
- **Small destruction in the target**
  - thin target
  - cf. $\pi^- \rightarrow \pi^0$
- **Ionization cooling**
  - suppress the emittance growth
- **Effective long target**



Effective target thickness

$$L_{eff} = N \times l$$

N : turn numbers

l : *thickness*